Pre-class Warm-up!!!

How many methods have we learned so far to solve differential equations?

a. 0
b. 1
c. 2
d. 3
e. ≥ 4
Section 1.6: Substitution methods and exact equations

We learn to recognize several kinds of differential equation:

- homogeneous equations
- Bernoulli equations
- Equations where some special substitution works
- Higher order equations where the order can be reduced
- Exact equations

The first 4 of these can be dealt with by introducing a new variable \( v \) dependent on both \( y \) and \( x \): \( v = g(x,y) \), where we can also write \( y = h(x,v) \). We then get \( dy/dx \) in terms of \( dv/dx \).

Exact equations are different!

Compare the methods we already learned:

- \( y' = \) function of \( x \)
- separate the variables
- first order linear equations
Homogeneous equations

Page 74 question 8 (like page 60 example 2)
Find the general solution to

\[ 2xy + x^2 y' = y^2 \]

A homogeneous equation is of the form

\[ \frac{dy}{dx} = f \left( \frac{y}{x} \right) \]

In this case: divide both sides by \( x^2 \)

\[ \frac{dy}{dx} + 2 \frac{xy}{x^2} = \frac{y^2}{x^2} \]

\[ \frac{dy}{dx} = \left( \frac{y}{x} \right)^2 - 2 \left( \frac{y}{x} \right) \]

Method: Put \( v = \frac{y}{x} \)

so

\[ y = vx, \quad \frac{dy}{dx} = \frac{d(vx)}{dx} = v + x \frac{dv}{dx} \]

Substitute

\[ v + x \frac{dv}{dx} = v^2 - 2v \]

\[ x \frac{dv}{dx} = v^2 - 3v \]

\[ \frac{dv}{dx} = \frac{v^2 - 3v}{x} \]

We can solve this by:

* integrating \( y' = \) function of \( x \)
* separating the variables
* as a first order linear equation
Page 74 question 8 (like page 60 example 2)
Find the general solution to
\[ 2xy + x^2 y' = y^2 \]

Summary:
\[ \nu = \frac{y}{x}, \quad x \frac{d\nu}{dx} = \nu^2 - 3\nu \]

Solve this by separating the variables.

\[ \int \frac{d\nu}{\sqrt{\nu^2 - 3\nu}} = \int \frac{dx}{x} \]

Partial fractions (calculation not shown)
\[ \int \frac{1}{3} \left( \frac{1}{\nu - 3} - \frac{1}{\nu} \right) d\nu = \int \frac{dx}{x} \]

\[ \frac{1}{3} \left( \ln|\nu - 3| - \ln|\nu| \right) = \ln x + C \]

\[ \ln \left( \frac{\nu - 3}{\nu} \right)^{\frac{1}{3}} = \ln x + C \]

\[ \left( \frac{\nu - 3}{\nu} \right)^{\frac{1}{3}} = e^{\ln x + C} = Bx \quad \text{where} \quad B = e^C \]

\[ \frac{\nu - 3}{\nu} = B^3 x^3 \]

\[ \nu - 3 = B^3 x^3 \nu \]

\[ \nu \left( 1 - B^3 x^3 \right) = 3 \]

\[ \nu = \frac{3}{1 - B^3 x^3} = \frac{y}{x} \]

\[ y = \frac{3x}{1 - B^3 x^3} \]
Bernoulli equations

1.6 question 23.
Find the general solution to

\[ xy' + 6y = 3x \ y^{4/3} \]

Form of a Bernoulli equation:

\[ y' + P(x) \ y = Q(x) \ y^n \]

Method: Substitute 

\[ v = y^{1-n} \]

In this case: 

\[ n = \frac{4}{3} \], put 

\[ v = y^{1 - \frac{4}{3}} = y^{-\frac{1}{3}} \]

\[ y = v^{-3} \]

\[ \frac{dy}{dx} = \frac{dv}{dx} - \frac{3}{v^4} \frac{dv}{dx} \]

\[ x \left( \frac{-3 v^{-4}}{3} \right) \frac{dv}{dx} + 6v^{-3} = 3x \ y^{-4} \]

Multiply by \( \frac{v^4}{3x} \):

\[ \frac{dv}{dx} - \frac{2v}{x} = -1 \]

\[ \frac{dv}{dx} - \frac{2v}{x} = -1 \] is first order linear.

Integrating factor: 

\[ e^{\int \frac{-2}{x} \, dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2} \]

\[ \frac{1}{x^2} \frac{dv}{dx} - \frac{2v}{x^3} = -\frac{1}{x^2} \]

\[ \frac{d}{dx} \left( \frac{v}{x^2} \right) = -\frac{1}{x^2} \]

\[ \frac{v}{x^2} = \frac{1}{x} + C \]

\[ v = x + Cx^2 = y^{-\frac{1}{3}} \]

\[ y = (x + Cx^2)^{-3} \]
Special substitutions

Questions from Section 1.6: solve the equations

16. \( y' = \sqrt{x+y+1} \)
28. \( x e^y y' = 2(e^y + x^3 e^{2x}) \)

Solution to 16. We are going to make a substitution \( v = \) something. What should it be?

a. \( v = e^{something} \)

b. \( v = y^{something} \)

c. \( v = \frac{y}{x} \)

d. \( v = x + y + 1 \)

e. \( v = \sqrt{x+y+1} \) \( \checkmark \)

\[
x + y + 1 = v^2, \quad y = v^2 - x - 1
\]
\[
\frac{du}{dx} = 2v \frac{dv}{dx} - 1 \quad \text{Substitute:}
\]
\[
2v \frac{dv}{dx} - 1 = \sqrt{v}
\]
\[
2v \frac{dv}{dx} = \sqrt{v} + 1
\]
\[
\frac{dv}{dx} = \frac{\sqrt{v} + 1}{2v} \quad \text{Separate the variables.}
\]
\[
\int \frac{2v \frac{dv}{dx}}{\sqrt{v} + 1} = \int dx
\]
\[
\int \frac{2v + 2 - 2}{\sqrt{v} + 1} dv = \int \left(2 - \frac{2}{\sqrt{v} + 1}\right) dv = 2v - 2\ln(\sqrt{v} + 1)
\]
\[
= \int dx = x + C
\]
\[
2\sqrt{x+y+1} - 2\ln(1+\sqrt{x+y+1}) = x + C
\]
gives a solution implicitly.
Questions from Section 1.6: solve the equation

28. \( x e^y y' = 2 \left( e^y + x^3 e^{2x} \right) \)

What substitution should we make?

a. \( v = e^y + x^3 e^{2x} \)
b. \( v = e^y \)
c. \( v = x e^y \)
d. \( v = x^3 e^{2x} \)
e. \( v = \)  

Try \( v = e^y \), \( y = \ln v \)

\[ \frac{dy}{dx} = \frac{1}{v} \frac{dv}{dx} \] Substitute.

We can solve this

a. by separating the variables
b. as a first order linear equation
   \( \checkmark \)
c. by making another substitution
Reducing the order of a differential equation

Section 1.6 question 44. Solve \( y y'' = (y')^2 \)

Solution: it’s significant that there is no term in \( x \) in the equation.

Substitute: \( v = \frac{dy}{dx} \)

\[ y'' = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = v \frac{dv}{dy} \]

Substitute in: \( y v \frac{dv}{dy} = v^2 \)

Divide by \( y v \).

\[
\frac{dv}{dy} - \frac{1}{y} v = 0
\]

I.F. \( e \int -\frac{1}{y} dy = e^{-\ln y} = \frac{1}{y} \)

\[ \frac{1}{y} \frac{dv}{dy} - \frac{1}{y^2} v = \frac{d}{dy} \left( \frac{v}{y} \right) = 0 \]

\[ \frac{v}{y} = C, \quad v = Cy = \frac{dy}{dx} \]

• We can solve this
  a. by separating the variables
  b. as a first order linear equation
  c. by making another substitution

\[ y = Be^{Cx} \text{ with two constants: } B, C \]
Section 1.6 question 46: Solve \( x y'' + y' = 4x \)

This time it’s significant there is no term in \( y \).

Put \( v = \frac{dy}{dx} = y' \)

\[
y'' = \frac{dv}{dx}
\]

Substitute: \( x \frac{dv}{dx} + v = 4x \)

We can solve this

\[
\frac{dv}{dx} + \frac{1}{x} v = 4
\]

a. by separating the variables

\( \checkmark \) b. as a first order linear equation

c. by making another substitution

Then solve the equation for \( v \) that arises.
Pre-class Warm-up!!!

How would you solve the following equation?

\[ 2xy + x^2 y' = y^2 \]

a. by integrating \( y' = \text{function of } x \)
b. separate the variables
c. as a first order linear equation \( v = \frac{y}{x} \)
d. as a homogeneous equation \( y' + 2v = v^2 \)
e. as a Bernoulli equation \( y' + \frac{2}{x} y = \frac{y^2}{x^2} \)
f. make a special substitution
g. reduce the order

What about \( xy^2 + 3y^2 - x^2 y' = 0 \)

\[ xy' + 2y = 6x^2 \sqrt{y} \]
\[ y' = \sqrt{x+y} \]
Question: Solve \[ 2xy \frac{dy}{dx} + y^2 = 10x \]

Solution: This equation is

\[ \frac{d}{dx}(xy^2) = 10x \]

Thus \[ xy^2 = \int 10x \, dx = 5x^2 + C \]

\[ y^2 = \frac{5x^2 + C}{x} \]

\[ y = \sqrt{5x + \frac{C}{x}} \]

How do we find a function \( f(x,y) \) so that

\[ \frac{df}{dx} = 2xy \frac{dy}{dx} + y^2 \]

Notice that

\[ \frac{df}{dx} = \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial x} \]

We solve \( f_y = 2xy \), \( f_x = y^2 \)

Thus

\[ f = xy^2 + g(x) \]

\[ f = xy^2 + h(y) \]

\[ f = xy^2 \text{ works.} \]

\( xy^2 \) is a potential function for the vector field

\[ \begin{bmatrix} y^2 \\ 2xy \end{bmatrix} = \nabla(xy^2) \]

It can look better to write the equation in differential form:

\[ 2xy \, dy + y^2 \, dx = 10 \, dx \]

\[ d(xy^2) = 10 \, dx \]
\[(2x^2 y + 1) \frac{dy}{dx} + 2xy^2 = 2x\]

The left side is \( \frac{df}{dx} \) where \( f \) is

a. \( x^2y + y \)

b. \( x^2y^2 + 1 \)

c. \( x^2y^2 + y \)

d. \( xy^2 + 2x^2y \)

e. None of the above.