Section 4.5: Row and column spaces

New vocabulary:
• row space, column space of a matrix
• row rank, column rank, rank
• pivot column
• null space

We learn:
• an algorithm to find a basis for the column space
• an algorithm to find a basis for the row space
• how to find a subset of a spanning set that is a basis
• how to extend an independent set to a basis
Find a subset of the vectors
\[
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}, \begin{bmatrix}
2 \\
4 \\
-2
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
4 \\
3 \\
-1
\end{bmatrix}
\]
that is a basis for the subspace they span.

Overview of the algorithm.

Take the first vector. If it is non-zero, keep it.

Is the second vector dependent on the vectors we have so far (i.e., a scalar multiple of the first)?

If dependent, throw it out.

If not, keep it.

Is the 3rd vector dependent on the first two?

Yes — throw it.

No — keep it.

We end with a set of vectors that is independent with the same span as the original vectors.
Find a subset of the vectors that is a basis for the subspace they span.

\[
\begin{bmatrix}
1 & 2 \\
2 & 4 \\
-1 & -2
\end{bmatrix}
\begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
4 \\
3 \\
-1
\end{bmatrix}
\]

We can solve the equation for \( x_1 \), it is consistent. The second vector is dependent on the first. We throw it. Is vector 3 dependent on 1st 2?

No. The row 0 0 -5 shows the eqns are inconsistent, keep vector 3.

More reduction, \( \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

Is vector 4 dependent on first 3?

Yes. Throw it. \( \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) are a basis for this space.

Solution: Is vector 2 dependent on vector 1?

Solve \( x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \)

Reduce \( \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix} \)

\( \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 3 & 3 \end{bmatrix} \)

\( \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)
Definition: the column space of a matrix $A$ is the span of the columns of the matrix.

Example: the column space of
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{bmatrix}
\]
is the space spanned by the vectors
\[
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}, \begin{bmatrix}
2 \\
4 \\
-2
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
4 \\
3 \\
-1
\end{bmatrix}
\]

Definition: a pivot column of a matrix $A$ is a column of $A$ for which the echelon form of $A$ has a leading entry (or pivot).

Example: The pivot columns of
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{bmatrix}
\]
are cols 1 & 3.

Theorem.
The pivot columns of a matrix $A$ form a basis for the column space of $A$.

Example: The vectors \[\begin{bmatrix}
\frac{1}{2} \\
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
\frac{3}{2} \\
0 \\
1
\end{bmatrix}\] are a basis for the column space of $A$.

Questions:
1. What is the dimension of the column space of the matrix $A$ in the example?
   a. 2
   b. 3
   c. 4
   a. 2  The basis has 2 elements.
2. Of what space is the column space of $A$ a subspace?
   a. $\mathbb{R}^2$
   b. $\mathbb{R}^3$
   c. $\mathbb{R}^4$
What is the dimension of the column space of the following matrix?

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

a. 0  
b. 1  
c. 2  
d. 3  
e. 4

When did we learn the meaning of the word `basis` for a vector space \( V \)?

a. This week  
b. Last week

c. 2  ✓
Definition. The row space of a matrix $A$ is the span of the rows of $A$.

Example. The row space of

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{pmatrix}$$

is the subspace of $\mathbb{R}^4$ spanned by

$$\begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix}, \begin{pmatrix}
2 \\
4 \\
-2
\end{pmatrix}$$

Look at the row space of

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{pmatrix}$$

(It's quicker to write down!)

It is the set of all vectors

$$a \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} + b \begin{pmatrix}
3 \\
4 \\
2
\end{pmatrix}$$

where $a, b \in \mathbb{R}$

Put it in echelon form

$$\begin{pmatrix}
1 & 2 \\
0 & -2
\end{pmatrix}$$

This has row space all vectors

Theorem 2. The row space of $A$ is not changed by elementary row operations. Hence $A$ has the same row space as its echelon form.

$$c \begin{pmatrix}
1 \\
2
\end{pmatrix} + d \begin{pmatrix}
0 \\
-2
\end{pmatrix}$$

These two sets of vectors are the same because

$$\begin{pmatrix}
0 \\
-2
\end{pmatrix} = \begin{pmatrix}
3 \\
4
\end{pmatrix} - 3 \begin{pmatrix}
1 \\
2
\end{pmatrix}$$

so

$$c \begin{pmatrix}
1 \\
2
\end{pmatrix} + d \begin{pmatrix}
0 \\
-2
\end{pmatrix} = c \begin{pmatrix}
1 \\
2
\end{pmatrix} + d \begin{pmatrix}
3 \\
4
\end{pmatrix}$$

Similarly we can write

$$a \begin{pmatrix}
1 \\
2
\end{pmatrix} + b \begin{pmatrix}
3 \\
4
\end{pmatrix}$$

as a linear combination of $\begin{pmatrix}
1 \\
2
\end{pmatrix}$, $\begin{pmatrix}
0 \\
-2
\end{pmatrix}$ because

$$\begin{pmatrix}
3 \\
4
\end{pmatrix} = \begin{pmatrix}
0 \\
-2
\end{pmatrix} + 3 \begin{pmatrix}
1 \\
2
\end{pmatrix}$$

$$a \begin{pmatrix}
1 \\
2
\end{pmatrix} + b \begin{pmatrix}
3 \\
4
\end{pmatrix} = a \begin{pmatrix}
1 \\
2
\end{pmatrix} + b \left( \begin{pmatrix}
0 \\
-2
\end{pmatrix} + 3 \begin{pmatrix}
1 \\
2
\end{pmatrix} \right)$$
Question: true or false in general?:

‘Let $A$ be a matrix. The column space of $A$ is not changed by elementary row operations. Hence $A$ has the same column space as its echelon form.’

True $\checkmark$  False
Theorem 2. The row space of $A$ is not changed by elementary row operations. Hence $A$ has the same row space as its echelon form.

Question like Section 4.5, 1-12

Find a basis for the row space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$$

Solution. Put it in echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows are now a basis for the row space.

They span (row space is unchanged).

They are independent: if $a \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -5 & -5 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & -5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$a, 2a, 3a - 5b, 4a - 5b = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$a = 0, \ 3a - 5b = 0, \ b = 0$.

This row space has dimension 2 as a subspace of $\mathbb{R}^4$.

Theorem 2 extra: The non-zero rows of the echelon form of $A$ form a basis for the row space of $A$. 

Definition. The row rank of a matrix $A$ is the dimension of its row space.

The column rank of a matrix $A$ is the dimension of its column space.

Theorem. For any matrix $A$, the row rank and column rank are equal.

Example. For the matrix

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{pmatrix}$$

They are both 2.

Definition. The common value of the row rank and column rank of a matrix is called the rank of the matrix.

Proof of theorem:

Row rank = number of non-zero rows in the echelon form.

Column rank = number of leading entries in the echelon form.

These are the same.
Question like Section 4.5, 17-20.

Find a basis for $\mathbb{R}^3$ that contains the vectors

\[
\begin{bmatrix}
3 \\
2 \\
-1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}
\]

Solution. (Check these vectors are independent.)

Adjoin

\[
\begin{bmatrix}
3 \\
2 \\
-1
\end{bmatrix},
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

to get a spanning set for $\mathbb{R}^3$.

Find an independent subset. Reduce

\[
\begin{bmatrix}
3 & 2 & 1 & 0 & 0 \\
2 & -2 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
2 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
3 & 2 & 1 & 0 & 0
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Leading entries in cols 1, 2, 4.

\[
\begin{bmatrix}
3 \\
2 \\
-1
\end{bmatrix},
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

is a basis for $\mathbb{R}^3$. 

\[
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Definition. The **nullspace** of $A$ is the vector space of solutions to $Ax = 0$.

It might be written $\text{Null } A$.

It is called the **nullity** of $A$.

Theorem. For any $m \times n$ matrix $A$,

$$\text{rank } A + \text{dim Null } A = n$$

Example. Find a basis for the nullspace of

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{bmatrix}
$$

They add to the total number of columns.

**Echelon form**

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & -5 & -5 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

**Free variables** $x$ and $z$

$$
y = -z \quad w = -2x - 3y - 4z = -2x - 2z
$$

**General solution**

$$
\begin{bmatrix}
w \\ x \\ y \\ z
\end{bmatrix} = \begin{bmatrix}
-2x - 2z \\ x \\ -z \\ 2
\end{bmatrix}
$$

$$
= x \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix} + z \begin{bmatrix}
0 \\
0 \\
-1 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
-2 \\
-1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
-1 \\
0
\end{bmatrix}
$$

is a basis.

Question: Have we done this calculation before?

Yes  No
Definition. The nullspace of $A$ is the vector space of solutions to $Ax = 0$. It might be written $\text{Null } A$. It is called the nullity of $A$.

Theorem. For any $m \times n$ matrix $A$, $\text{rank } A + \text{dim Null } A = n$

Example. Find a basis for the nullspace of

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 \\
-1 & -2 & 0 & -1
\end{bmatrix}
$$
Questions:

Are the following true or false in general for an $n \times n$ matrix $A$?

a. If $A$ has rank $n$ then $Ax = b$ has a solution for every vector $b$ in $\mathbb{R}^n$.

True  False  Not enough information to say.

b. If $Ax = 0$ has a unique solution then $Ax = b$ always has a solution, for every $b$ in $\mathbb{R}^n$.

True  False  Not enough information to say.
Question: Suppose that $A$ is an $m \times n$ matrix ($m$ rows, $n$ columns) and that $Ax = 0$ has a unique solution. Which of the following statements is sometimes false?

a. The columns of $A$ are linearly independent.

b. The columns of $A$ span a space of dimension $n$.

c. The columns of $A$ are a basis for the space they span.

d. $m \leq n$