Let $A$ be the matrix $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$.

Which of the following vectors $v$ have the property that $Av$ is a scalar multiple of $v$?

a. $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c. $v = \begin{bmatrix} 1 \\ i \end{bmatrix} \checkmark$

d. None of the above vectors

\[ \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 21 \\ 25 \end{bmatrix} \neq 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \text{No} \]

\[ \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 25 \\ 31 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{Yes} \]
6.1 Eigenvalues and eigenvectors

New vocabulary:
• eigenvalues and eigenvectors
• the characteristic polynomial of a matrix

We learn:
• how to find the eigenvalues and eigenvectors as the result of a theorem about the characteristic polynomial

What we don’t learn:
• why we should be interested in e-values and e-vectors
Definition of an eigenvector and eigenvalue of an $n \times n$ matrix $A$.

We say that a vector $v$ is an eigenvector of $A$ with eigenvalue $\lambda$ if $Av = \lambda v$ and $v \neq 0$. $\lambda$ is a number.

Example: Let $A = 
\begin{bmatrix}
3 & 4 \\
4 & 3
\end{bmatrix}$.

Try $v = 
\begin{bmatrix}
1 \\
1
\end{bmatrix}$

$Av = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 
\begin{bmatrix} 7 \\ 7 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ with e-value 7.

Try $v = 
\begin{bmatrix}
1 \\
-1
\end{bmatrix}$

$Av = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 
\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with e-value $-1$.

Definition of the characteristic polynomial of $A$.

This is the polynomial $\text{det}(A - \lambda I)$ where $\lambda$ is a variable and $I$ is the $n \times n$ identity matrix.

Example: Let $A = 
\begin{bmatrix}
3 & 4 \\
4 & 3
\end{bmatrix}$. The characteristic polynomial is $\text{det}\begin{bmatrix} 3 - \lambda & 4 \\ 4 & 3 - \lambda \end{bmatrix} = (3 - \lambda)(3 - \lambda) - 4 \cdot 4 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$

Its roots are 7, -1.
Theorem. For an $n \times n$ matrix $A$, the eigenvalues of $A$ are precisely the solutions of the characteristic equation

$$\det(A - \lambda I) = 0$$

Proof

To find the eigenvalues of $A$: find the roots $\lambda$ of the characteristic polynomial.

To find the eigenvectors of $A$:

We solve $Av = \lambda v$ i.e. we solve $(A - \lambda I)v = 0$.

i.e. we compute the nullspace of $(A - \lambda I)$.

Definition: the eigenspace for the eigenvalue $\lambda$ is defined to be $\text{Null}(A - \lambda I)$.

The numbers $\lambda$ that appear as $e$-values are precisely the solutions to the characteristic equation.
To find the eigenvalues of $A$: We solve
$$\det(A - \lambda I) = 0$$
To find the eigenvectors of $A$: find the eigenvalues. For each $\lambda$-value, solve $Av = \lambda v$, $N(A - \lambda I)v = 0$. Find a basis for $\text{Null}(A - \lambda I)$

Question like 6.1, 1-26.
Find the eigenvalues and eigenvectors of
$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

Solution: Step 1. The characteristic polynomial is $\lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$
The $\lambda$-values are $7, -1$

Step 2. We solve $(A - \lambda I)v = 0$.

\[ \lambda = 7 : \quad A - \lambda I = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \]
To find the nullspace: Echelon form
$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}$$ Basis for nullspace: \[ \begin{bmatrix} 1 \end{bmatrix} \]

\[ \lambda = -1 : \quad A - \lambda I = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \quad \text{Echelon form} \]
$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$$ Basis for nullspace \[ \begin{bmatrix} -1 \end{bmatrix} \]

We obtain two $\lambda$-values, 7, -1, with $v$-vectors \[ \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix} \] respectively.
6.1 question 21.
Find the eigenvalues and eigenvectors. Find a basis for each eigenspace of dimension 2 or larger.

\[ A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \]

Find the eigenspaces:

\[ \lambda = 2 \]

Find the nullspace of

\[ A - 2I = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

Echelon form

\[ \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Two free variables.

Basis for the \( \lambda = 2 \) e-space

\[ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \]

For \( \lambda = 1 \) we get an e-vector

\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Solution

Char. poly: \[ \det \begin{bmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda) \det \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = (2-\lambda)((4-\lambda)(\lambda-3)-1+6) = (2-\lambda)(\lambda^2-3\lambda+2) = -\lambda^3+5\lambda^2-8\lambda+4 \]

Find the roots. Try to find some root \( \lambda = 0 \) ? No \( \lambda = 1 \) Yes!! \( \lambda - 1 \) is a factor

\[ \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 \]
Question: For the identity matrix \( I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

1. How many distinct eigenvalues does \( I \) have? \( 1 \) occurs with mult. 3, Answer 1.

2. How many independent eigenvectors can we find for \( I \)? Answer 3.

a. 0
b. 1
c. 2
d. 3
e. 4 or more

Another question: Same questions 1 and 2 for the matrix \( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \).