Pre-class Warm-up!!!

True or False?

The matrix \[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\] has 3 eigenvectors, no two of which are scalar multiples of each other.

a. True

b. False

It has up to scalar multiple only the eigenvectors \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) with eigenvalues 2 and 3.

Do you remember last time we did a lot with the matrix \[
\begin{bmatrix}
3 & 4 \\
4 & 3
\end{bmatrix}
\]?

It has eigenvectors \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) with eigenvalues 7, -1.
6.2 Diagonalization of matrices

New vocabulary:
• diagonalize, diagonalizable, similar

We learn:
• the connection between eigenvalues, eigenvectors and diagonalization
• how to diagonalize a matrix (when it is diagonalizable).
• Some theorems: a criterion for diagonalizability; independence of eigenvectors when eigenvalues are distinct; distinct eigenvalues implies diagonalizable.

What we don’t really learn:
• why we would want to diagonalize matrices

Definition. Square matrices $A$ and $B$ are similar if there is an invertible matrix $P$ so that
$$B = P^{-1} A P$$

A square matrix $A$ is diagonalizable if it is similar to a diagonal matrix. So:
$$D = P^{-1} A P$$ is diagonal for some invertible $P$. 

Definition. Square matrices $A$ and $B$ are similar if there is an invertible $n \times n$ matrix $P$ so that $P^{-1}AP = B$.

A square matrix $A$ is diagonalizable if it is similar to a diagonal matrix.

Example: $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Try the matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Now the columns of $P$ are the e-vectors of $A$ and the diagonal entries of $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ are the corresponding e-values.

Calculate $P^{-1}AP = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 14 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$

is diagonal.
Theorem. Let $A, P, D$ be $n \times n$ matrices with $P$ invertible and $D$ diagonal. Then $P^{-1}AP = D$ if and only if the columns of $P$ are eigenvectors for $A$ with eigenvalues the diagonal entries in $D$.

Proof. $P^{-1}AP = D \iff AP = PD$

Write the columns of $P$ as $v_1, \ldots, v_n$.

$P = [v_1 | v_2 | \ldots | v_n]$. Write $D = [\lambda_1 \quad 0 \quad \ldots \quad 0 \quad 0]^T$.

$AP = PD$ means: for each $i$, $Av_i = \lambda_i v_i$, $i^{th}$ column of $AP$.

$\lambda_i v_i = i^{th}$ column of $PD$.

$e.g. \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} [1 \\ -1] = \begin{bmatrix} 7 & -1 \\ 7 & 1 \end{bmatrix} [1 \\ -1] = [7 \\ -1]$.

$\iff$ for each $i$, $Av_i = \lambda_i v_i$.

$\iff$ for each $i$, $v_i$ is an eigenvector of $A$ with $e$-value $\lambda_i$. \[\square\]
Example. The matrix \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) is not diagonalizable.

Proof. We show \( A \) does not have 2 independent e-vectors. Find e-values:

Characteristic poly: \( \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 \) Roots: 1 (twice).

To find e-vectors: find \( \text{Null}(A-I) \)

\( = \text{Null} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \)

1 free variable. \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is a basis for the nullspace. There is one e-vector up to scalar multiple. \( A \) is not diagonalizable.

Also \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \) are not diagonalizable.

Like 6.2 questions 1-28
Find whether or not the following matrices are diagonalizable. If so, find \( P \) so that \( P^{-1}AP = D \) is diagonal.

1. \( A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \)
2. \( A = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix} \)

Solution: char poly \((1-\lambda)(3-\lambda)\). Two e-values

1. find nullspace of \( A-I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \)

It has basis \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

3. \( \text{Null} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \) has basis \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

Take \( P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Then \( P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \)
Like 6.2 questions 1-28
Find whether or not the following matrices are diagonalizable. If so, find \( P \) so that \( P^{-1}AP = D \) is diagonal.

1. \( A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \)
2. \( A = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix} \)

Two more matrices:
3. \( A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix} \)
4. \( A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \)

Solution 2.

\[ \text{Char. poly} = \det \begin{bmatrix} -5-\lambda & -12 \\ 3 & 7-\lambda \end{bmatrix} \]
\[ = \lambda^2 + 5\lambda - 7\lambda - 35 + 36 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \]

\( \lambda = 1 \) is the only root, twice.

Find e-vectors: \( \text{Null } A - I = \text{Null } \begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix} \)

Echelon form \( \begin{bmatrix} -6 & -12 \\ 0 & 0 \end{bmatrix} \). Basis for nullspace \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). There is only one e-vector up to scalar multiple. \( A \) is not diagonalizable.

Question: How many distinct eigenvalues does matrix 4. have? If you get to it: is it diagonalizable?

a. 0
b. 1
c. 2
Theorem 2. If $A$ has eigenvectors $v_1, \ldots, v_k$ associated to distinct eigenvalues, then $v_1, \ldots, v_k$ are independent.

Proof. Let the eigenvalues be $\lambda_1, \ldots, \lambda_k$ so $Av_i = \lambda_i v_i$ for each $i$.

Suppose $v_1, \ldots, v_k$ were dependent.

Let $c_1v_1 + \ldots + c_kv_k = 0$ be a non-zero dependence relation. Reorder the $v_i$ and let $c_1v_1 + \ldots + c_r v_r$ be a shortest such relation. Apply the matrix $A - \lambda_r I$.

\[(A - \lambda_r I) v_i = Av_i - \lambda_r v_i = \lambda_i v_i - \lambda_r v_i = 0 \text{ if } i = r,\]

\[\neq 0 \text{ if } i \neq r,\]

We get a shorter relation if $r > 1$.

This is a contradiction. The $v_1, \ldots, v_k$ are independent.

Theorem 3. If the $n \times n$ matrix $A$ has $n$ distinct eigenvalues, it is diagonalizable.
Page 354 Question 32
Show that if \( n \times n \) matrices \( A \) and \( B \) are similar, then they have the same characteristic equation, and therefore have the same eigenvalues.

Page 354 Question 29
Prove: if the matrices \( A \) and \( B \) are similar and the matrices \( B \) and \( C \) are similar, then the matrices \( A \) and \( C \) are similar.