Pre-class Warm-up!!!

Which of the following is the slope field for the equation \( \frac{dy}{dx} = \frac{x}{y} \)?

- a.
- b. Ignore IIII
- c. Ignore the vertical lines
- d. Ignore II

Another question. Which of these is the direction field for \( x' = y, \ y' = x \)?

- Also b.
7.1 First order linear systems of equations

We learn:
• what is a linear system of equations
• how to convert a high order differential equation into a system of first order equations
• occasionally, how to convert a system of equations into a higher order differential equation
• a theorem about existence of solutions to systems of equations
• what are direction fields

Vocabulary:
• direction field, solution curve or trajectory, phase plane portrait
• homogeneous means the same thing that it did before, except in a new context.
Page 372 question 13:

Solve \( x' = -2y, \ y' = 2x, \ x(0) = 1, \ y(0) = 0. \)

Implicit in this is that \( x, y \) are functions of \( t \).

Solution: From eq. 1: \( x'' = -2y' \)

From eq. 2: \( x'' = -4x, \ x'' + 4x = 0 \)

Char. poly: \( r^2 + 4 = 0 \), roots \( \pm 2i \)

Solution: \( x = A \cos 2t + B \sin 2t \)

\( x' = -2A \sin 2t + 2B \cos 2t = -2y \)

\( y = A \sin 2t - B \cos 2t \).

\[
\begin{bmatrix}
 x \\
 y
\end{bmatrix} = A \begin{bmatrix}
 \cos 2t \\
 \sin 2t
\end{bmatrix} + B \begin{bmatrix}
 \sin 2t \\
 -\cos 2t
\end{bmatrix}
\]

When \( t = 0 \) \( \begin{bmatrix}
 x \\
 y
\end{bmatrix} = \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = A \begin{bmatrix}
 1 \\
 0
\end{bmatrix} + B \begin{bmatrix}
 0 \\
 -1
\end{bmatrix} \)

so \( B = 0, \ A = 1 \)

\[
\begin{bmatrix}
 x \\
 y
\end{bmatrix} = \begin{bmatrix}
 \cos 2t \\
 \sin 2t
\end{bmatrix}
\]

The system is homogeneous if \( F = \begin{bmatrix}
 f_1 \\
 f_n
\end{bmatrix} = \begin{bmatrix}
 0 \\
 0
\end{bmatrix} \)

A linear system of differential equations has the form

\[
\begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
\end{bmatrix}' = \begin{bmatrix}
 p_{11} & \cdots & p_{1n} \\
 \vdots & \ddots & \vdots \\
 p_{n1} & \cdots & p_{nn}
\end{bmatrix} \begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
\end{bmatrix} + \begin{bmatrix}
 f_1 \\
 f_2 \\
 \vdots \\
 f_n
\end{bmatrix}
\]

where \( x_i, \ p_{ij}, \ f_i \) are all functions of \( t \).

Example:

\[
\begin{bmatrix}
 x \\
 y
\end{bmatrix} = \begin{bmatrix}
 0 & -2 \\
 2 & 0
\end{bmatrix} \begin{bmatrix}
 x \\
 y
\end{bmatrix} + \begin{bmatrix}
 0 \\
 0
\end{bmatrix}
\]

Write this as \( X' = PX + F \)

Solutions are vector functions of \( t \).

The \( p_{ij} \) need not be linear functions of \( t \).
Page 371 question 2:
Transform the given differential equation into an equivalent system of first order differential equations.
\[ x'''' + 6x'' - 3x' + x = \cos 3t \]

Solution:
Write \( x_1 = x, \ x_2 = x_1', \ x_3 = x_2' \)
\( x_4 = x_3', \ x_4' = -6x'' + 3x' - x + \cos 3t \)
This is
\[
\begin{bmatrix}
  x_1' \\
  x_2' \\
  x_3' \\
  x_4'
\end{bmatrix} = \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 \\
  -1 & 3 & -6 & 0
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} + \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \cos 3t
\end{bmatrix}
\]

Page 371 question 10:
Same with \( x'' = (1-y)x, \ y'' = (1-x)y \)
(Note: this system won’t be linear)

Solution
\[ x_1 = x, \ x_2 = x_1', \ y_1 = y, \ y_2 = y_1' \]
Section 1.6 question 46: Solve \( x y'' + y' = 4x \)

This time it’s significant there is no term in \( y \).

Put \( v = \frac{dy}{dx} = y' \)

\[ y'' = \frac{dv}{dx} \]

Substitute: \( x \frac{dv}{dx} + v = 4x \).

We can solve this

a. by separating the variables

\[ \frac{dv}{dx} + \frac{1}{x} v = 4 \]

b. as a first order linear equation

c. by making another substitution

Then solve the equation for \( v \) that arises.

Question: Recall 1.6 question 46 above!
What does the equation \( xy'' + y' = 4x \) look like when we write it as a linear system?

a. \[ \begin{bmatrix} v' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{v}{x} \\ 4 \end{bmatrix} \]

b. \[ \begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} -\frac{v}{x} \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \]

d. None of the above.
Page 372 question 19:
Find the solution, and draw a direction field for
\( x' = -y, \ y' = 13x + 4y; \ x(0) = 0, \ y(0) = 3. \)

The slope of each vector's
\( \frac{y'}{x'} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} \)

A curve following the vectors is called a solution curve or trajectory.

Solution:
\( x'' = -y' = -13x - 4y = -13x + 4x' \)

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = 
\begin{bmatrix}
    -e^{-2t} \sin 3t \\
    e^{-2t} 3 \cos 3t + 2 \sin 3t
\end{bmatrix}
\]
Theorem 1.
Given a first order system \( X' = PX + F \) and a vector \( B \), if the functions \( P \) and \( F \) are continuous around some number \( t = a \), then there is a unique solution satisfying \( X(a) = B \).