**Math 8302** **Homework 11** **PJW**

**Date due: Wednesday April 17, 2002.**

1. Let $f$ and $g$ be $C^1$-maps $M \to \mathbb{R}$ where $M$ is a smooth manifold, and let $P \in M$. Show that
$$d(f \cdot g)_P = g \cdot (df)_P + f \cdot (dg)_P.$$

2. Let $u : S^1 \to \mathbb{R}^2$ be the vector field on $S^1$ described geometrically as the map $u : S^1 \to \mathbb{R}^2$ given by $u(x,y) = (-y,x)$, so that $u(x,y)$ is always perpendicular to $(x,y)$. Let $v(x,y) = y \cdot u(x,y)$ and $w(x,y) = x \cdot u(x,y)$.
(a) Calculate $[u,v]$ and $[v,w]$.
(b) Show that if $X$ is a vector field on $S^1$ for which $[u,X] = 0$ then $X$ is a scalar multiple of $u$.

3. Let $TS^1$ be the tangent bundle of $S^1$. Prove that $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$.

4. (a) Let $f : M \to W$ be a $C^1$ map of manifolds and suppose that $y = f(x)$ is a regular value of $f$, so that $f^{-1}(y)$ is a submanifold of $M$. We will regard the tangent bundle $Tf^{-1}(y)$ as a submanfold of $TM$. Show that with this identification, at each point $P \in f^{-1}(y)$ we have $T_P f^{-1}(y) = \mathrm{Ker}(df)_P$, where $(df)_P : T_P M \to T_y(W)$ is the map induced by $f$ on the tangent spaces.
(b) Regarding the tangent space $T_A(M_n(\mathbb{R}))$ as the space of all $n \times n$-matrices, for each matrix $A$, show that $T_I SL(n,\mathbb{R})$ is the set of matrices of trace 0, and $T_I O(n,\mathbb{R})$ is the set of skew-symmetric matrices, where $I$ is the identity matrix.

**Extra Question (do not hand in)**

5. Verify that if $X$, $Y$, and $Z$ are vector fields then $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$.