Math 8302 **Homework 12** **PJW**

Date due: Wednesday April 24, 2002. There will be a quiz on this date.

1. In this question we consider the manifold $M = \mathbb{R}$ with its usual differentiable structure.
   (a) Show that there is no flow $\xi : \mathbb{R} \times M \to M$ with $\xi(1, P) = P^3$ for all $P \in M$.
   (b) Let $\gamma : \mathbb{R} \to M$ be the curve $\gamma(t) = t^3$ and for each $t_0 \in \mathbb{R}$ let $X_{\gamma(t_0)} \in T_{\gamma(t_0)}M$ be the velocity vector $X_{\gamma(t_0)} := \dot{\gamma}_{t_0}$ of γ at t_0. Show that $P \mapsto X_P$ is not a smooth vector field on M. Show that there is no smooth vector field on M for which γ is an integral curve.

2. Let $M = \mathbb{R}^2$ with usual coordinates x^1 and x^2, and let $X = x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2}$. Calculate the integral curve to X through an arbitrary point $(a^1, a^2) \in \mathbb{R}^2$, and draw the flow lines. Show that X determines a flow (not just a local flow) ξ and write down an explicit formula for $\xi_t(a^1, a^2) = \xi(t, (a^1, a^2))$. Verify that your formula satisfies

$$\xi_{t_1+t_2}(a^1, a^2) = \xi_{t_1}(\xi_{t_2}(a^1, a^2)).$$

3. Let X be a smooth vector field on M and suppose that it generates a flow $\xi : \mathbb{R} \times M \to M$. Prove that for each $P \in M$,

$$(d\xi_t)_P(X_P) = X_{\xi_t(P)}.$$

4. Let M be a manifold for which TM is diffeomorphic to the product vector bundle $M \times \mathbb{R}^n$ (i.e. M is parallelizable as on page 51 of Aubin's book). Suppose that M is compact. Show that there is a diffeomorphism $M \to M$ which has no fixed points and which is homotopic to the identity map. Deduce that S^2 is not parallelizable.

   [This is similar to a very standard type of exam question, e.g. Fall 1996 no. 3, Fall 1998 no. 7b (almost the same question!). In these questions you may be asked also to show that the Euler characteristic $\chi(M)$ is zero. We have not adequately covered the material to do this. In any case, those exam questions seem to ask you only to state the result. One approach to this is to use the Lefschetz number $L(f)$ of a map $f : M \to M$, described in Rotman's book. It has the properties
   (i) $L(1) = \chi(M)$,
   (ii) if f and g are homotopic then $L(f) = L(g)$, and
   (iii) if f has no fixed points then $L(f) = 0$.]

Extra questions

Exercises 3.27, 3.30, 3.31, 3.33 from Aubin's book.
Manifolds and Topology exam: Spring 97 No. 5.