Math 8302 **Homework 2** **PJW**

Date due: Wednesday February 6, 2002. There will be a quiz on this day.

1. Let K be a finite simplicial complex with vertices $\{v_1, \ldots, v_n\}$. We may triangulate the cone on K as an (abstract) simplicial complex CK whose vertices are $\{v_1, \ldots, v_n, a\}$, where a is a new vertex, and in which the simplices have vertex sets $\{a\}$, $\{v_{i_0}, \ldots, v_{i_d}\}$ and $\{v_{i_0}, \ldots, v_{i_d}, a\}$ whenever $\{v_{i_0}, \ldots, v_{i_d}\}$ is a simplex of K. We may define the *(unreduced) suspension* ΣK of K as the (abstract) simplicial complex whose vertices are $\{v_1, \ldots, v_n, a, b\}$, where a and b are two new vertices, and in which the simplices have vertex sets $\{a\}$, $\{b\}$, $\{v_{i_0}, \ldots, v_{i_d}\}$, $\{v_{i_0}, \ldots, v_{i_d}, a\}$ and $\{v_{i_0}, \ldots, v_{i_d}, b\}$ whenever $\{v_{i_0}, \ldots, v_{i_d}\}$ is a simplex of K. Pictures which indicate what ΣK looks like are given on pages 232 and 329 of Rotman.
   (i) Show that if $X = \{0, 1\}$ consists of just two points (with the discrete topology) then $\Sigma(\Sigma(X))$ is the triangulation of the 2-sphere which looks like an octahedron.
   (ii) For an arbitrary finite simplicial complex X show that $H_n(\Sigma X) \cong H_{n-1}(X)$ if $n \geq 2$. (For $n = 1$ we have $H_1(\Sigma X) \cong \tilde{H}_0(X)$ — do not prove this.)
   (iii) For an arbitrary finite simplicial complex X show that $\chi(\Sigma X) = 2 - \chi(X)$.

2. By considering a suitable triangulation of S^n, show that $\chi(\mathbb{R}P^n) = \frac{1}{2}(1 + (-1)^n)$.

3. Let X and Y be 2-manifolds. (See page 195.) The *connected sum* of X and Y is the space $X \# Y$ obtained by deleting a small open disc from each of X and Y and identifying the boundaries of the holes which this produces.
   (i) Show that $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$.
   (ii) 2-manifolds may be constructed by identifying edges of polygons in pairs. Thus the torus, projective plane and Klein bottle may be constructed from a square using the identifications

   Identify each of the following two spaces by name (giving reasons):

   (iii) Calculate the Euler characteristic of the surface: