Math 8302 **Homework 3** **PJW**

Date due: Wednesday February 13, 2002.

1. In defining the oriented simplicial homology of a simplicial complex K we initially put the vertices of K in some order and define $C_n(K)$ to have basis the symbols $[v_0, \ldots, v_n]$ where $v_0 < \cdots < v_n$ and $\{v_0, \ldots, v_n\}$ are the vertices of a simplex of K. The boundary operator has only been defined on such symbols. Subsequently we have given meaning to other orderings of the symbols as follows:

$$[v_{\pi(0)}, \ldots, v_{\pi(n)}] = (-1)^{\mathrm{sign}(\pi)}[v_0, \ldots, v_n]$$

   where π permutes $\{0, \ldots, n\}$. Prove that

$$d([v_{\pi(0)}, \ldots, v_{\pi(n)}]) = \sum_{i=0}^{n}(-1)^i[v_{\pi(0)}, \ldots, \hat{v}_{\pi(i)}, \ldots, v_{\pi(n)}]$$

   by following steps 1 and 2 below. A hat over a symbol means that the symbol has been omitted. (It may help to do e.g. $d[v_1, v_0, v_2]$ to see what is going on here. Note that the left hand side of the equation is defined as $(-1)^{\mathrm{sign}(\pi)}d[v_0, \ldots, v_n] = (-1)^{\mathrm{sign}(\pi)}\sum(-1)^i[v_o, \ldots, \hat{v}_i, \ldots, v_n]$.)

   Step 1. Prove the formula in the case when $\pi = (i, i+1)$.
   Step 2. Show that the result for arbitrary π follows from Step 1.

2. Without cutting up the hexagon and arranging it into something more convenient, calculate all the homology groups of the surface:

3. 7.32 on page 164 of Rotman.

4. Let M be a surface obtained from a polygon by identifying the edges in pairs. Using the words 'similar pair' and 'reversed pair' as in the lecture, prove that $M \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ for some number of factors if and only if there is a similar pair of edges on the boundary of the polygon, and that M is a connected sum of tori if and only if all pairs of edges are reversed.