Math 8302 **Homework 7** **PJW**

Date due: Wednesday March 13, 2002.

1. Let $g : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the mapping $g(A) = A^T A$. Show that g has rank $\frac{n(n+1)}{2}$ at all matrices A which satisfy $g(A) = 1$.

2. Let $\mathbb{R}$ be the real line with the standard differential structure and let X be a copy of the real line with differential structure defined by an atlas consisting of a single chart $\phi : X \to \mathbb{R}$ where $\phi(t) = t^3$.
   (a) Is the set-theoretic identity map from X to $\mathbb{R}$ a diffeomorphism?
   (b) Are X and $\mathbb{R}$ the same differential manifold?
   (c) Are X and $\mathbb{R}$ diffeomorphic?

3. Let $K \subseteq W \subseteq \mathbb{R}^n$, where K is closed and W is open.
   (a) Show that there is a C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = 1$ if $x \in K$ and $f(x) = 0$ if $x \notin W$.
   (b) Let $g : W \to \mathbb{R}$ be a continuous function. Prove that the restriction $g|_K$ extends to a continuous function $h : \mathbb{R}^n \to \mathbb{R}$.

4. = Question 1.43 on page 39 of Aubin. Show that a proper injective immersion is an embedding.

5. Let N be a smooth submanifold of M and suppose that $f : X \to N$ is a mapping with the property that the composite $\iota \circ f$ is smooth, where $\iota : N \to M$ is the inclusion mapping. Show that f is a smooth mapping.

Further questions (do not hand these in):

A. Let $GL(n, \mathbb{R})$ denote the set of invertible $n \times n$ matrices with entries in $\mathbb{R}$, an open subset of the set of all $n \times n$ matrices.
   (a) Let $\alpha \in GL(n, \mathbb{R})$. Show that there is a number $K > 0$ so that whenever β is a matrix with $\|\beta\| < K$, then $\alpha + \beta$ is invertible, with inverse given by the formula

   $$(\alpha + \beta)^{-1} = \alpha^{-1} - \alpha^{-1}\beta\alpha^{-1} + \alpha^{-1}\beta\alpha^{-1}\beta\alpha^{-1} - \cdots.$$

   You should determine in terms of α a value of K which works.
   (b) Let $\iota : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ be the inversion map $\iota(\alpha) = \alpha^{-1}$. Prove that ι is differentiable at every $\alpha \in GL(n, \mathbb{R})$, with derivative $\iota'(\alpha)(\beta) = -\alpha^{-1}\beta\alpha^{-1}$.

B. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Show that the set of points of $\mathbb{R}^n$ at which rank $f < n$ is a closed subset. (This was used in passing in the proof of the inverse function theorem on page 10 line 4.)

C. The space of $M_n(\mathbb{R})$ of $n \times n$-matrices is a vector space of dimension n^2 which has as a basis the matrices E_{ij} which are zero except for a 1 in position (i, j). Let x^{ij} denote the corresponding coordinate variable. Let $\det : M_n(\mathbb{R}) \to \mathbb{R}$ be the determinant function.

For each (i,j) compute the partial derivative $\frac{\partial}{\partial x^{ij}}\det$ evaluated at the identity matrix I. Use the implicit function theorem to show that the matrices with determinant 1 can be parametrized near I as

$$\begin{pmatrix} g(t_{12},\ldots,t_{nn}) & t_{12} & t_{13} & \cdots \\ t_{21} & t_{22} & \cdots & \\ \vdots & & & \end{pmatrix}$$

for a certain function g. Calculate g' when $t_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$.

D. Let $p(t) = e^{\frac{1}{t-2}-\frac{1}{t-1}}$ for $1 \le t \le 2$ and $p(t) = 0$ for $t \notin (1,2)$. Put

$$\gamma(r) = \frac{\int_r^2 p(t)\,dt}{\int_1^2 p(t)\,dt}$$

as on page 26 of Aubin. Show that γ is a smooth function satisfying

$$\gamma(r) = \begin{cases} 1 & \text{if } r \le 1 \\ 0 & \text{if } r \ge 2 \end{cases}$$

and that $0 \le \gamma(r) \le 1$ for all r. Is γ analytic?

E. Show that the product of manifolds is a manifold. Show that the product of smooth maps is smooth. Show that the product of immersions is an immersion.

F. Let $f : M \to N$ be an immersion and let $W \subseteq M$ be a submanifold. Show that $f|_W : W \to N$ is an immersion.

G. = Question 1.42 on page 39 of Aubin. In part b) construct the map ψ by taking as its coordinate functions the various mappings f_i and also the mappings $\Omega_i \to \mathbb{R}^n$ given by the chart functions. This is a completely standard result which is proved in various texts. For example it appears as Theorem 1.5.7 in the book by Conlon.

H. (a) Define $f : S^2 \to \mathbb{R}^4$ by the formula

$$f(x,y,z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2).$$

Prove that f passes to a well-defined embedding $\bar{f} : \mathbb{R}P^2 \to \mathbb{R}^4$.
(b) Show that there are six points $p_1,\ldots,p_6 \in \mathbb{R}P^2$ such that

$$g : \mathbb{R}P^2 - \{p_1,\ldots,p_6\} \to \mathbb{R}^3$$

specified by $g(x,y,z) = (yz, xz, xy)$ is an immersion. The mapping g of $\mathbb{R}P^2$ into $\mathbb{R}^3$ is known as Steiner's surface.