Math 8302 **Homework 8** **PJW**

Date due: Wednesday March 27, 2002. There will be a quiz on this date.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^1$ be the mapping defined by $f(x, y, z) = x^2 + y^2 - z^2$.
   (a) Show that the subset $f^{-1}(a)$ of $\mathbb{R}^3$ is a manifold if $a \neq 0$, and is not a manifold if $a = 0$.
   (b) Prove that if a and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. [Hint: consider scalar multiplication by $\sqrt{b/a}$ on $\mathbb{R}^3$.]
   (c) Extra: do not hand in. Pictorially examine the catastrophic change in the topology of $f^{-1}(c)$ as c passes through the critical value.
2. Let $f : M \to N$ be a smooth map between manifolds. Show that $\{(x, f(x)) \mid x \in M\}$ is a submanifold of $M \times N$, and that it is diffeomorphic to M.
3. Let $f : M \to N$ be an embedding of manifolds. Show that $f(M)$ is a submanifold of N.
4. Let $f : M \to N$ be a submersion and let U be an open subset of M. Show that $f(U)$ is an open subset of N.
5. (a) If M is compact and N connected, show that every submersion $f : M \to N$ is surjective.
   (b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

Extra questions: do not hand in.

A. Suppose that $L \subseteq M \subseteq N$ are submanifolds, and $x \in L$. Show that there exist independent functions $g_1, \ldots, g_l$ on a neighborhood W of z in N such that $L \cap W = \{y \in W \mid g_1(y) = 0, \ldots, g_l(y) = 0\}$ and $M \cap W = \{y \in W \mid g_1(y) = 0, \ldots, g_m(y) = 0\}$, where $l - m$ is the codimension of L in M.

B. Show that the curve $t \mapsto (t, t^2, t^3)$ embeds $\mathbb{R}^1$ into $\mathbb{R}^3$. Find two independent functions that globally define the image. Are your functions independent on all of $\mathbb{R}^3$, or just on an open neighborhood of the image?