Math 8201 **Final Homework** **PJW**

Date due: 4pm Wednesday December 18, 2002

There are six questions altogether. Give careful and complete arguments. You may quote without proof any results which appear in the book by Dummit and Foote, any results from homework questions which you have been assigned, and any results which were lectured in class, provided that you indicate that you are doing this and that **they do not invalidate the question** (in my opinion). No other results may be quoted. Relevant arguments found in any book may be used, but should not be copied word for word. The work should be your own – please do not consult other people. I will be available to give clarification of the meaning of questions, but I will not give anyone hints. You may find it convenient to contact me by email: webb@math.umn.edu; or my office telephone: (612) 625 3491; or my home telephone: (507) 645 8150.

A. (Spring 1994, question 5) (18%)
(i) (4) Let G be a finite group. Prove that the number of conjugates in G of a subgroup H equals the index of its normalizer $N_G(H)$ in G.
(ii) Let now G be a simple group of order $1092 = 4 \cdot 3 \cdot 7 \cdot 13$.
a) (4) Find the number of Sylow 13-subgroups and the number of Sylow 7-subgroups of G.
b) (5) Prove that G has a single conjugacy class of subgroups of index 14.
c) (5) Prove that G has no subgroup of index 13.
[You may assume Sylow's theorems.]

B. (Fall 1994, question 6) (18%)
Prove that a finite group is cyclic if and only if for every $k \geq 1$ it contains at most k elements of order dividing k.
[First prove the result for groups of prime power order and then use Sylow's theorems.]

C. (Spring 2000, question 1) (10%) Let G be a group whose order is a power of a prime number, say p^n. Prove for each i between 0 and n, that G has a subgroup of order p^i.

PLEASE TURN OVER.

D. (Spring 1995, question 7) (18%) For each prime number p put

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b\}$$

as a subring of $\mathbb{Q}$. Show that $\mathbb{Z}_{(p)}$ has a unique non-zero prime ideal. Show further that if $p \neq q$ are distinct primes then $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ has just two non-zero prime ideals, each of which is maximal.

E. (Spring 1998, question 5) (18%) Let R be a commutative ring (with 1), and let R^* be its group of invertible elements. Assume that R has only finitely many maximal ideals and that R^* is finite. Prove that R is finite.

F. (Spring 2001, question 6) (18%) Let K be a field.
  (a) (6) Show that for every element $a \in K$, the rings $K[X]/(X^2)$ and $K[X]/((X-a)^2)$ are isomorphic.
  (b) (12) Let A be any ring which contains K as a subring, and suppose that as a vector space over K, $\dim(A) = 2$. Show that either $A \cong K[X]/(X^2)$ or $A \cong K \times K$ or A is a field.