Math 8201 **Homework 4** **PJW**

Date due: October 7, 2002

There will be a 30 minute quiz in class on this date on the subject matter of Homeworks 3 and 4. Hand in only the starred questions.

Section 3.1 5, 14*, 36, 37, 40, 41*

F. Let H be a subgroup of G that contains the commutator subgroup G' of G. Prove that $H \triangleleft G$.

G*. Prove that if N is a normal subgroup of G such that G/N is abelian then $N \supseteq G'$, the commutator subgroup.

H. Let N be a normal subgroup of G and let g be an element of G of finite order. Show that the order of the element Ng of G/N divides the order of g. Suppose now that N has index 2 in G. Show that all the elements of G which do not lie in N have even order.

I. Let H be the group of rotations of the tetrahedron. Show that H has no subgroup of order 6.

Section 3.2 4, 21, 22, 23

J*. Show that if H and K are subgroups of G such that $H \supseteq K$ and $|G : K|$ is finite, then $[G : K] = [G : H][H : K]$.

K. Let H_1 and H_2 be subgroups of G. Show that any left coset relative to $H_1 \cap H_2$ is the intersection of a left coset of H_1 with a left coset of H_2 Use this to prove *Poincaré's Theorem* that if H_1 and H_2 have finite index in G then so has $H_1 \cap H_2$.

L. Show that if A is a subgroup of G of index 2 then for any subgroup H of G, $|H : H \cap A|$ equals 1 or 2.

Section 3.3 3, 7, 9

M. Let $H \triangleleft G$ and let $\pi : G \to G/H$ be the natural map. Suppose that X is a subset of G so that $\pi(X)$ generates G/H. Prove that $G = \langle H \cup X \rangle$.

N. Let G be a finite group with a normal subgroup H such that $(|H|, |G : H|) = 1$. Show that H is the unique subgroup of G having order $|H|$.
[Hint: If K is another such subgroup, what happens to K in G/H?]

O. If $H \triangleleft G$, need G contain a subgroup isomorphic to G/H?

P. Let p be a prime and let H and K be subgroups of a finite G, each of which has order a power of p, and such that H is normal in G.
(a) Show that HK is a subgroup of G whose order is a power of p.

(b) Suppose in addition that K is normal in G (so now both H and K are normal in G). Show that HK is normal in G.

(c) Show that G has a unique largest normal subgroup whose order is a power of p, and that this subgroup contains all other normal subgroups whose order is a power of p. (This subgroup is often denoted $O_p(G)$.)

(d) Show that the factor group $G/O_p(G)$ has no normal subgroup of order a power of p, apart from the identity subgroup.

Q. (a) Let G be a group of order 24 which has a normal subgroup H of order 8. Show that every element of G not in H has order divisible by 3.
(b) Determine $O_2(S_4)$.

R*. Let G be the dihedral group of order 12, which we may regard as the group of isometries of a regular hexagon. Let $\sigma \in G$ be the rotation through an angle of 180° about the midpoint of the hexagon. We have seen in class that $\langle \sigma \rangle$ is the center of G, and hence is a normal subgroup.
(a) Show that $G/(\sigma) \approx S_3$.
(b) Make a complete list of all subgroups H with $(\sigma) \subseteq H \subseteq G$. For each possible order that H can have, specify how many subgroups there are of that order.

S*. (Amplification of Sec. 4.4 no. 1.)An automorphism of a group G is said to be *inner* if it has the form $x \mapsto axa^{-1}$ for some $a \in G$, in which case we might write I_a for this automorphism.
(a) Show that the assignment $a \mapsto I_a$ is a homomorphism $G \to \operatorname{Aut} G$. Deduce that the set of inner automorphisms is a subgroup of $\operatorname{Aut} G$. This subgroup is denoted $\operatorname{Inn} G$
(b) Show that the kernel of this homomorphism is the center $Z(G)$ of G, and deduce that $\operatorname{Inn} G \cong G/Z(G)$.
(c) Prove that $\operatorname{Inn} G$ is a normal subgroup of $\operatorname{Aut} G$. The factor group $\operatorname{Aut} G/\operatorname{Inn} G$ is called the *group of outer automorphisms.*