Math 8202 **Homework 12** **PJW**

Date due: April 28, 2003. There will be a quiz on this date.

Hand in only the starred questions. This week we are studying parts of Sections 14.4, 14.6 and 14.7. We are omitting Sections 14.5 and 14.8 which in my view give a level of detail beyond the requirements of this course. In Section 14.4 the main thing we will do is the theorem of the primitive element. It is not worth spending a lot of time on the first part of Section 14.4, but it is basic and you should probably either be able to work out for yourself the results given, or quickly see that they are true. Skim the start of 14.4. From Section 14.6 I will give only an overview of the discriminant. We will not study symmetric functions (although they are on the syllabus). The detailed recipe for what to do with a quartic on page 596 is interesting, but certainly not worth memorizing. You may read the algebraic proof of the Fundamental Theorem of Algebra, but we will not study it. From Section 14.7 we will do only the implication that if a polynomial can be solved by radicals then it has a solvable Galois group, and also the example on page 610. The converse implication uses the norm and Hilbert's Theorem 90, which we will not study. Also, we will not go into detail with Cardano's formulas.

Section 14.4, page 576 1, 2, 3*, 4*, 5*.
Section 14.6, page 598 11, 12, 18, 19, 20, 47*, 48, 49.
Section 14.7, page 616 12*, 13

N. (Spring 1994, qn. 2)
Let f be an irreducible cubic polynomial in $\mathbb{Q}[X]$. The discriminant d of f may be defined by

$$-d = (\xi_1 - \xi_2)^2(\xi_1 - \xi_3)^2(\xi_2 - \xi_3)^2$$

where ξ_1, ξ_2 and ξ_3 are the roots of f in a splitting field.
(i) (5) Using the above definition, show that d belongs to $\mathbb{Q}$.
(ii) (5) Let $r = (\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_2 - \xi_3)$. Show that if the Galois group of f over $\mathbb{Q}$ is S_3, the symmetric group of degree 3, then r does not lie in $\mathbb{Q}$. Hence deduce that $-d$ cannot be expressed as a square in $\mathbb{Q}$ (i.e. for no element $x \in \mathbb{Q}$ is it true that $x^2 = -d$).
(iii) (6) Show conversely that if $-d$ cannot be expressed as a square in $\mathbb{Q}$ then the splitting field for f has even degree over $\mathbb{Q}$, and deduce in this case that the Galois group of f must be S_3.