Math 8202 **Homework 4** **PJW**

Date due: February 24, 2003. There will be a quiz on this date.

Hand in only the starred questions.

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D*. (Modification of Fall 1993, qn. 8) Let M be the subgroup of $\mathbb{Z}^3$ generated by the three vectors $(2, 4, 4)$, $(6, 3, -6)$ and $(4, 14, 20)$.
 (a) Calculate the rank of M.
 (b) Calculate the invariant factors and the elementary divisors of $\mathbb{Z}^3/M$.
 (c) Find a basis f_1, f_2, f_3 for $\mathbb{Z}^3$ with the property that $a_1 f_1, \ldots, a_r f_r$ is a basis for M, where r is the rank of M, and where $a_1 \mid \cdots \mid a_r$.

E*. Let $A = \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$.
 (a) Calculate the invariant factors of A.
 (b) Calculate the elementary divisors of A.
 (c) Calculate the structure of the group $3A/9A$.

The following is a collection of past exam questions that are relevant for the material we are now covering. Some of them use ideas (notably the idea of a projective module) which we have not yet done. These questions are included here only for your information – you are not asked to do any of them!

2. (Spring 1999) (a) (9 pts) Let A be an $n \times n$ matrix with integer entries. Regarding the free abelian group $\mathbb{Z}^n$ as the set of column vectors of length n with integer entries, let H be the subgroup of $\mathbb{Z}^n$ generated by the columns of A. Prove that the group $\mathbb{Z}^n/H$ is finite if and only if $\det A \neq 0$.
 (b) (5 pts) Give an example of two subgroups of the group $\mathbb{Z} \oplus \mathbb{Z}$ each of which is a direct summand of $\mathbb{Z} \oplus \mathbb{Z}$ but such that their sum is not a direct summand of $\mathbb{Z} \oplus \mathbb{Z}$. Give reasons for your assertions.

2. (Spring 2001) (14%) Let R be a commutative ring, $L = R^n$ a free R-module of rank n, and $A \in M_n(R)$ an $n \times n$ matrix viewed as an endomorphism of L.
 (a) (5) Show that $\det(A) \cdot L \subseteq \mathrm{Im}(A)$.
 (b) (9) If $R = \mathbb{Z}$ and $\det(A) \neq 0$, show that the size of $\mathrm{Coker}(A)$ equals $|\det(A)|$.

3. (Fall 2001) (11%) (a) (7) Let A be a finitely generated abelian group with a subgroup B with the property that whenever $na \in B$ for some $n \in \mathbb{Z}$ and $a \in A$ then $a \in B$. Show that $A \cong B \oplus A/B$.
 [Additive notation is being used for these groups, so that na means $a + a + \cdots + a$ added n times. You may assume the structure theorem for finitely generated abelian groups.]
 (b) (4) Let D be the subgroup of the free abelian group $C = \mathbb{Z}^3$ generated by the vector $(10, 6, 14)$. Show that C is not isomorphic to $D \oplus (C/D)$.

3. (Spring 2002) (15%) Let A be a finitely generated abelian group, let B be a subgroup and put $C = A/B$. Suppose that

$$\begin{aligned} A &= \mathbb{Z}^u \oplus F_A, \\ B &= \mathbb{Z}^v \oplus F_B, \\ C &= \mathbb{Z}^w \oplus F_C, \end{aligned}$$

where F_A, F_B and F_C are finite abelian groups.

(a) (9%) Show that $u = v + w$.
[If you use properties of the tensor product, they should be proved. You may assume the Structure Theorem for finitely generated abelian groups.]

(b) (6%) Suppose further that $F_C = 0$. Show that $F_B = F_A$.

3. (Fall 2002) (14%) Let $A = \mathbb{Z}^3$ be a free abelian group of rank 3, and let B be the subgroup of A generated by the elements $(2, -4, -1)$, $(4, 1, 1)$ and $(-2, -2, 1)$ (where we regard elements of A as row vectors of length 3 with integer entries). Writing

$$A/B = \mathbb{Z}^t \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z},$$

calculate the values of the integers $t, d_1, \ldots, d_s$.