O. (Fall 1993, no. 7) (14%)
Let A be a unitary $n \times n$ matrix with real entries.
(a) (7%) Show that every eigenvalue of A has absolute value 1.
(b) (7%) Assuming further that n is odd, show that A has an eigenvalue equal to $\det A$.

P. (Spring 1994, no. 6) (12%)
Let A be a real $m \times n$ matrix. Regarding A as the matrix of a linear map $\mathbb{R}^n \to \mathbb{R}^m$, let V be a subspace of $\mathbb{R}^n$ so that $\mathbb{R}^n = V \oplus \operatorname{Ker} A$. Show that the bilinear form $\langle\ ,\ \rangle$ defined on V by

$$\langle u, v \rangle = u^T A^T A v \quad \text{for all } u, v \in V$$

is non-singular, where ^{T} denotes the transpose. Hence show that the matrices A and $A^T A$ have the same rank.

Q. (Fall 1994, no. 7) (15%)
Let V be a real finite-dimensional vector space with an inner product (x, y), and let α be an endomorphism of V. Establish the existence of an endomorphism α' such that

$$(x, \alpha y) = (\alpha' x, y)$$

for all $x, y \in V$.
Show that the non-zero vectors x such that $(\alpha x, \alpha y) = 0$ for all y satisfying $(x, y) = 0$ are the eigenvectors of the endomorphism $\alpha' \alpha$.

R. (Fall 2000, qn. 4) (15%) Let k be a field.
a) (8) Prove that every finitely generated projective $k[X]$-module is isomorphic to $k[X]^n$ for some n.
b) (7) Prove that every submodule of the regular representation of $k[X]$ is projective.
[You may assume the structure theorem for finitely generated modules over a principal ideal domain.]

S. (Fall 2001, qn. 3) (11%) (a) (7) Let A be a finitely generated abelian group with a subgroup B with the property that whenever $na \in B$ for some $n \in \mathbb{Z}$ and $a \in A$ then $a \in B$. Show that $A \cong B \oplus A/B$.
[Additive notation is being used for these groups, so that na means $a + a + \cdots + a$ added n times. You may assume the structure theorem for finitely generated abelian groups.]
(b) (4) Let D be the subgroup of the free abelian group $C = \mathbb{Z}^3$ generated by the vector $(10, 6, 14)$. Show that C is not isomorphic to $D \oplus (C/D)$.

T. (Fall 2001, qn. 8) (12%) (a) (8) Let $(\ \ ,\ \ )$ be the bilinear form on $\mathbb{R}^3$ specified on column vectors $\mathbf{u}$ and $\mathbf{v}$ by $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\mathbf{u}^T$ is the transpose of the vector $\mathbf{u}$. Determine whether or not this bilinear form is positive definite.

(b) (4) Give an example of a symmetric bilinear form on a real vector space which is not positive definite, and such that there is a basis $e_1, \ldots, e_n$ for the space with $(e_i, e_i) > 0$ for every i.

U. (Spring 2002, qn. 3) (15%) Let A be a finitely generated abelian group, let B be a subgroup and put $C = A/B$. Suppose that

$$A = \mathbb{Z}^u \oplus F_A,$$
$$B = \mathbb{Z}^v \oplus F_B,$$
$$C = \mathbb{Z}^w \oplus F_C,$$

where F_A, F_B and F_C are finite abelian groups.

(a) (9%) Show that $u = v + w$.
[If you use properties of the tensor product, they should be proved. You may assume the Structure Theorem for finitely generated abelian groups.]

(b) (6%) Suppose further that $F_C = 0$. Show that $F_B = F_A$.

V. (Fall 2002, qn. 4) (11%) Let R be a ring with a 1 (which we denote 1_R) and P a projective left R-module.

(a) (3%) Show that P is necessarily a unital R-module, that is, $1_R \cdot x = x$ for all $x \in P$.
[For this question we do not assume as an axiom that all modules are unital.]

(b) (8%) Suppose that I is a 2-sided ideal in R and that, when regarded as a left R-module, I is projective. Show that IP is projective as a left R-module.