Review for the final exam

The exam is on Thursday 12/15/05 from 1:30-4:30 in room Vincent Hall 211. I will hold a review session in our usual lecture room from 10:00-12:00 on Thursday morning. After that I will go bowling with whomever will come with me in the basement of Coffman Union. I will buy pizzas for you there, but perhaps you could pay for your own bowling. Even if you don't want to bowl you can still eat pizza.

There are 12 questions on the exam, with each question part usually worth 6% of the total. You may not use books or notes. You may use a calculator. Always show your work, and be sure to write down sufficient detail so that I can see that you are able to do all calculations without a calculator if necessary. If you are not sure what is required in any question, or what the question means, do ask.

1. Let $f : \mathrm{Mat}(2,2) \to \mathbb{R}$ be the mapping $f(A) = \mathrm{trace}(A^2)$. Find the directional derivative of f at the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in the direction of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2. Consider the following functions which are defined to be 0 at $(0,0)$. Are they continuous at the origin? Differentiable? Do the partial derivatives exist?

$$\frac{x^2}{\sqrt{x^2+y^2}}, \qquad \frac{2x-5y}{\sqrt{x^2+y^2}}, \qquad \frac{xy}{\sqrt{x^2+y^2}}, \qquad \frac{x^2+y^2}{x+y^2}$$

3. True or false? For each of the following statements, decide whether it is true or false, and then either give brief reasons or a counterexample to justify your assertion.
   (a) There exists a surjective linear mapping $\mathbb{R}^7 \to \mathbb{R}^{10}$.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $gf = 1_{\mathbb{R}^2}$, the identity mapping on $\mathbb{R}^2$.
   (c) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $fg = 1_{\mathbb{R}^3}$, the identity mapping on $\mathbb{R}^2$.
   (d) Let $v_1, \ldots, v_r$ be a linearly independent set of vectors in a vector space V and $w_1, \ldots, w_r$ another set of vectors in a vector space W. Then there exists a linear mapping $T : V \to W$ with $T(v_i) = w_i$ for all i with $1 \le i \le r$.
   (e) If S is an $m \times n$ matrix of rank m then there exists an $n \times m$ matrix T with $ST = I$, the identity matrix.
   (f) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and there exists $g : \mathbb{R}^n \to \mathbb{R}^n$ with $fg = gf = 1$. Then g is differentiable.
   (g) If $f : U \to V$ and $g : V \to W$ are linear mappings then $\mathrm{rank}(gf) \le \mathrm{rank}(f)$ always.
   (h) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $ST = I$.

(i) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $TS = I$.

(j) If $S : U \to V$ is a linear mapping which is 1-1 then there exists a linear mapping $T : V \to U$ with $ST = I$.

(k) If $S : U \to V$ is a linear mapping which is 1-1 then there exists a linear mapping $T : V \to U$ with $TS = I$.

4. Find the number of paths of length 4 from vertex A to itself in the graph

5. Let S be a subset of $\mathbb{R}^n$. We will say that x is a *limit point* of $S \Leftrightarrow$ for all $\epsilon > 0$ there exists $y \in S$ with $0 < |x - y| < \epsilon$. Using the definition that S is closed $\Leftrightarrow$ for every point x not in S there is a ball of some positive radius with center x which contains no point of S, prove that

$$S \text{ is closed} \Leftrightarrow S \supseteq \text{its limit points}.$$

Which of the following statements means x is not a limit point of S?

(i) There exists $\epsilon > 0$ such that for all $y \in S$ either $y = x$ or $|y - x| \geq \epsilon$.

(ii) There exists $\epsilon > 0$ such that for all $y \in S$ either $y = x$ or $|y - x| > \epsilon$.

(iii) There exists $\epsilon > 0$ such that there exists $y \in S$ with either $y = x$ or $|y - x| \geq \epsilon$.

(iv) There exists $y \in S$ such that there exists $\epsilon > 0$ with either $y = x$ or $|y - x| > \epsilon$.

6. Do one step of Newton's method to solve the system of equations

$$\begin{aligned} ye^x + xe^y &= 1 \\ x^3 + xy + \sin y &= 0 \end{aligned} \quad \text{starting at } a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

7. Calculate $\det \begin{pmatrix} 1 & 2 & -1 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}$.

8. Prove that

$$Df(a)(h) = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t}.$$

H&H Section 3.3: 1, 13, 14.
Section 3.4: 1, 2, 3, 4, 5.
Section 3.6: 1, 2, 6, 7, 8.
Section 3.9: 5, 17, 18, 19.