Math 8202 **Final Homework** **PJW**

Date due: 3pm Monday May 15, 2006

There are six questions altogether. Give careful and complete arguments. You may quote without proof any results which appear in the book by Dummit and Foote, any results from homework questions which you have been assigned, and any results which were lectured in class, provided that you indicate that you are doing this and that **they do not invalidate the question** (in my opinion). No other results may be quoted. Relevant arguments found in any book may be used, but should not be copied word for word. The work should be your own – please do not consult other people. I will be available to give clarification of the meaning of questions, but I will not give anyone hints. You may find it convenient to contact me by email: webb@math.umn.edu; or my office telephone: (612) 625 3491; or my home telephone: (507) 645 8150.

C. (Spring 2002, qn. 4) (17%) Let R be a commutative Noetherian ring with a 1 and I a maximal ideal of R.

(a) (9%) Show that if M is a finitely generated R-module then M/IM has finite composition length as an R-module.

(b) (8%) Show that R/I^{10} has finite composition length as an R-module.

K. (Spring 1995, qn. 3) (17%) Let $f(X) \in k[X]$ be a polynomial of degree n over k and let K be a splitting field for f over k. Suppose that the Galois group $\mathrm{Gal}(K/k)$ is the symmetric group S_n.

(a) (7) Show that f is irreducible in $k[X]$, and separable.

(b) (6) Let α be a root of f in K. Show that in $k(\alpha)[X]$, f factorizes as

$$f(X) = (X - \alpha)g(X)$$

where $g(X)$ is an *irreducible* polynomial.

(c) (4) Determine the Galois group of g over $k(\alpha)$.

L. (Fall 2002, qn. 6) (17%) Let a be a nonzero rational number.

(a) (6%) Determine the values of a such that the extension $\mathbb{Q}(\sqrt{ai})$ is of degree 4 over $\mathbb{Q}$, where $i^2 = -1$.

(b) (11%) When $K = \mathbb{Q}(\sqrt{ai})$ is of degree 4 over $\mathbb{Q}$ show that K is Galois over $\mathbb{Q}$ with the Klein 4-group as Galois group. In this case determine all the quadratic extensions of $\mathbb{Q}$ contained in K.

PLEASE TURN OVER.

P. (Spring 1994, no. 6) (17%)
Let A be a real $m \times n$ matrix. Regarding A as the matrix of a linear map $\mathbb{R}^n \to \mathbb{R}^m$, let V be a subspace of $\mathbb{R}^n$ so that $\mathbb{R}^n = V \oplus \operatorname{Ker} A$. Show that the bilinear form $\langle\ ,\ \rangle$ defined on V by

$$\langle u, v\rangle = u^T A^T A v \quad \text{for all } u, v \in V$$

is non-singular, where ^{T} denotes the transpose. Hence show that the matrices A and $A^T A$ have the same rank.

T. (Fall 2001, qn. 8) (16%) (a) (10) Let $(\quad,\quad)$ be the bilinear form on $\mathbb{R}^3$ specified on column vectors $\mathbf{u}$ and $\mathbf{v}$ by $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\mathbf{u}^T$ is the transpose of the vector $\mathbf{u}$. Determine whether or not this bilinear form is positive definite.
(b) (6) Give an example of a symmetric bilinear form on a real vector space which is not positive definite, and such that there is a basis $e_1, \ldots, e_n$ for the space with $(e_i, e_i) > 0$ for every i.

W. (Fall 2001) (16%) (a) (10) Let A be a finitely generated abelian group with a subgroup B with the property that whenever $na \in B$ for some $n \in \mathbb{Z}$ and $a \in A$ then $a \in B$. Show that $A \cong B \oplus A/B$.
[Additive notation is being used for these groups, so that na means $a + a + \cdots + a$ added n times. You may assume the structure theorem for finitely generated abelian groups.]
(b) (6) Let D be the subgroup of the free abelian group $C = \mathbb{Z}^3$ generated by the vector $(10, 6, 14)$. Show that C is not isomorphic to $D \oplus (C/D)$.