**Math 8245** **Homework 5** **PJW**

**Date due: November 13, 2006. Either hand it to me in class or put it in my mailbox by 3:30.**

1. (i) Show that $C_2 * C_2$ is isomorphic to the group of distance-preserving mappings $\mathbb{R} \to \mathbb{R}$ generated by the two mappings $\alpha$ and $\beta$ defined by $\alpha(x) = -x$ and $\beta(x) = -x + 2$.
(ii) Show that $C_2 * C_2$ is isomorphic to the group of 2 by 2 integer matrices generated by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.
(iii) Show that $C_2 * C_2$ has an infinite cyclic subgroup of index 2 and that every other element has order 2.
(iv) Show that every subgroup of $C_2 * C_2$ is isomorphic to $1, C_2, C_\infty$ or $C_2 * C_2$.
[You may use any techniques you wish to do this. You could use Bass-Serre theory to deduce the free product decomposition, or not!]

2. Let $N$ be kernel of the homomorphism $SL(2, \mathbb{Z}) \to C_{12} = \langle x \rangle$ which sends $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $x^3$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ to $x^2$. You may assume without proof that $N$ is a free group. Find a set of matrices which are free generators for $N$.

3. Let $Y$ be a connected graph with maximal subtree $Y_0$ and suppose that $A$ is a group with subgroups $A(v)$ and $A(e) \le A$ for each vertex $v \in Y$ and edge $e \in Y$. Suppose that $A(e) \le A(\iota e)$ always, and for each edge $e \in Y$ there is an element $a_e$ so that $a_e^{-1} A(e) a_e \subseteq A(\tau e)$. Suppose that $a_e = 1$ for all $e \in Y_0$. Let $\Gamma$ be the coset graph determined by this information, so that the vertices of $\Gamma$ are the cosets $gA(v)$ of the various groups $A(v)$, the edges are the cosets $gA(e)$ of the various groups $A(e)$ and we have $\iota gA(e) = gA(\iota e)$ and $\tau gA(e) = g a_e A(\tau e)$. Show that $\Gamma$ is connected if and only if the subgroups $A(v)$ and elements $a_e$ taken together generate $A$.
[It is completely acceptable to reduce this question to a result proven in class and quote that result. You should assume that in case some of the groups $A(v)$ or $A(e)$ happen to be the same, for different $v$ and $e$, then we take the vertices and edges of $\Gamma$ to be distinct sets in bijection with the sets of cosets $gA(v)$ and $gA(e)$, not the actual sets of cosets.]

4. With the set-up of the last question, let the group $A$ be $SL(2, \mathbb{Z})$, let $Y$ be a single edge with its two vertices, and let $A(\iota e) = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$, $A(\tau e) = \langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ and $A(e) = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ with $A_e = 1$. Prove that the coset graph $\Gamma$ of the last question is a tree.
[ It is completely acceptable to reduce this question to a result proven in class and quote that result. You may assume anything you want about the geometry of the action of $SL(2, \mathbb{Z})$ on the upper half plane.]

5. With the set-up of question 3, suppose that $Y$ has a single vertex with a number of edges which all start and finish at that vertex. Suppose that $A(v) = 1$. Show that each connected component of $\Gamma$ is isomorphic to the Cayley graph of the subgroup of $A$ generated by the elements $a_e$.

6. Let $(G(-), Y, Y_0)$ be a graph of groups in which $Y$ is a single edge $e$ with its end vertices, so there is an injective group homomorphism $t_e : G(e) \to G(\tau e)$. Let $\alpha : G(\tau e) \to G(\tau e)$ be a group automorphism and consider the graph of groups with exactly the same specification except that $t_e$ is replaced by $\alpha \circ t_e$. Show that the fundamental groups of these two graphs of groups are isomorphic.

7. Let
$$B = \mathbb{Z}[\frac{1}{3}] = \{\frac{a}{3^n} \mid a, n \in \mathbb{Z}\} \subseteq \mathbb{Q}.$$
Let $\theta : B \to B$ be the group automorphism $\theta(x) = x/3$ and define $A = B \rtimes \langle\theta\rangle$. Let $Y$ be a graph with a single vertex $v$ and a single edge $e$, which is a loop. Put $A(v) = A(e) = \mathbb{Z} \subseteq B$ and let $a_e = \theta$.
   (i) Consider the coset graph defined in question 3 and show that it is a tree.
   (ii) Find the cycle type of the action of a generator of $A(v)$ on the set of vertices of $\Gamma$ which are distance 2 from the vertex $A(v)$.
   (iii) Prove that $A$ can be expressed as an HNN extension with vertex and edge group $\mathbb{Z}$.

**Extra Questions**

8. Is $C_2 * C_2$ isomorphic to a subgroup of $SL(2, \mathbb{Z})$? Is $C_2 * C_2$ isomorphic to a subgroup of $PSL(2, \mathbb{Z})$?

9. Express the matrices $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$ as products of the generators $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ of $SL(2, \mathbb{Z})$.