Math 8246 **Homework 1** **PJW**
Date due: Monday February 5, 2007

1. (D&F 10.4, 4) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

2. (D&F 10.4, 5) Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

3. (D&F 10.4, 6) If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.

4. (D&F 10.4, 11) Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

5. (D&F 10.4, 16) Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.
   (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.
   (b) Prove that there is an R-module isomrohpsm $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.

6. Show that as a ring, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ is the direct sum of two fields. [You may assume the ring structure from Proposition 19 of D&F if you can't guess what it must be, and also question 25 from 10.4 of D&F.]

7. (D&F 10.5, 14(a)) Let $0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$ be a sequence of R-modules.
   (a) Prove that the associated sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\phi'} \mathrm{Hom}_R(D, N) \to 0$$

   is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [To show the sequence splits, take $D = N$ and show the lift of the identity map in $\mathrm{Hom}_R(N, N)$ to $\mathrm{Hom}_R(N, M)$ is a splitting homomorphism for ϕ.]
   (b) is a similar statement obtained by applying $\mathrm{Hom}_R(-, D)$ to the short exact sequence. Do not bother with this part of the question.

8. (D&F 10.5, 21) Let R and S be rings with 1 and suppose M is a right R-module, and N is an (R, S)-bimodule. If M is flat over R and N is flat as an S-module prove that $M \otimes_R N$ is flat as a right S-module.

9. (D&F 10.4, 15) Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \ldots$ Tensor products do commute with arbitrary direct sums – also an exercise you could do!]