Math 8246 **Homework 2** **PJW**

Date due: Monday February 19, 2007

1. (D&F 17.1, 8) Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}^n_R(N, D) \to \mathrm{Ext}^n_R(M, D) \to \mathrm{Ext}^n_R(L, D) \to 0$ is also short exact and split. [Use a splitting homomorphism and Proposition 5, which says that Ext is functorial in each variable.]

2. (D&F 17.1, 12) Prove that $\mathrm{Tor}^R_0(D, A) \cong D \otimes_R A$.

3. (D&F 17.1, 19) Suppose $r \neq 0$ is not a zero divisor in the commutative ring R.
   (a) Prove that multiplication by r gives a free resolution $0 \to R \xrightarrow{r} R \to R/rR \to 0$ of the quotient R/rR.
   (b) Prove that $\mathrm{Ext}^0_R(R/rR, B) = {}_rB$ is the set of elements $b \in B$ with $rb = 0$, that $\mathrm{Ext}^1_r(R/rR, B) \cong B/rB$, and that $\mathrm{Ext}^n_r(R/rR, B) = 0$ for $n \geq 2$ for evry R-module b.
   (c) Prove that $\mathrm{Tor}^R_0(A, R/rR) = A/rA$, that $\mathrm{Tor}^R_1(A, R/rR) = {}_rA$ is the set of elements $a \in A$ with $ra = 0$, and that $\mathrm{Tor}^R_n(A, R/rR) = 0$ for $n \geq 2$ for every R-module A.

4. (i) Suppose that A, B, and C are R-modules and that there are homomorphisms

$$A \underset{\delta}{\overset{\alpha}{\rightleftarrows}} B \underset{\gamma}{\overset{\beta}{\rightleftarrows}} C$$

   such that $\beta\alpha = 0$ and such that the identity map on B can be written $1_B = \alpha\delta + \gamma\beta$. Show that $\beta = \beta\gamma\beta$. Suppose in addition to all this that $\alpha = \alpha\delta\alpha$. Show that $B \cong \alpha\delta(B) \oplus \gamma\beta(B)$.

   (ii) A chain complex $\mathcal{C}$ of R-modules is called *contractible* if it is chain homotopy equivalent (by R-module homomorphisms) to the zero chain complex. Prove that $\mathcal{C}$ is contractible if and only if $\mathcal{C}$ can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where α is an isomorphism.

Given a homomorphism of chain complexes of R-modules $\phi : \mathcal{C} \to \mathcal{D}$ we may define $E_n = C_{n-1} \oplus D_n$, and a mapping $e_n : E_n \to E_{n-1}$ by $e_n(a, b) = (-\partial a, \phi a + \partial b)$, where we denote the boundary maps on $\mathcal{C}$ and $\mathcal{D}$ by ∂. The specification $\mathcal{E}(\phi) = \{E_n, e_n\}$ is called the *mapping cone* of ϕ.

5. Show that $\mathcal{E} = \{E_n, e_n\}$ is indeed a chain complex.

6. Show that there is a short exact sequence of chain complexes $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{C}[1] \to 0$ where $\mathcal{C}[1]$ denotes the chain complex with the same R-modules and boundary maps as $\mathcal{C}$ but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

$$\cdots \to H_n(\mathcal{C}) \to H_n(\mathcal{D}) \to H_n(\mathcal{E}(\phi)) \to H_{n-1}(\mathcal{C}) \to \cdots$$

Show that $\mathcal{E}(\phi)$ is acyclic if and only if ϕ induces an isomorphism $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ for every n.

7. Show that if $\phi \simeq \psi : \mathcal{C} \to \mathcal{D}$ then $\mathcal{E}(\phi) \cong \mathcal{E}(\psi)$.

8. Show that the two extensions $\mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z} \xrightarrow{\mu'} \mathbb{Z} \xrightarrow{\epsilon'} \mathbb{Z}/3\mathbb{Z}$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod{3}$ and $\epsilon'(1) \equiv 2 \pmod{3}$.

9. Let $0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0$ be a short exact sequence.
   (i) Construct its inverse under the group operation in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ with sufficient precision that you can determine by examination of the two sequences whether or not they are equivalent.
   (ii) Determine the isomorphism type of middle term of the sum of the sequence with itself. [By 'the sum' is meant the addition in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.]