Math 8211 **Homework 2** **PJW**

Date due: Monday October 15, 2012. In class on Wednesday September 17 we will grade your answers, so it is important to be present on that day, with your homework.

As practice, but not part of the homework, make sure you can do questions in Rotman apart from the ones listed below, such as 2.19a.

Assignment questions:
Rotman pages 64-69: 2.13, 2.18, 2.20 (assume without proof Proposition 2.42).
Rotman pages 94-97: 2.28, 2.34, 2.36(i)
Questions 1, 2, 3 below.

0. Preliminary facts for questions 1 and 2, to be thought about, but **not written down or handed in**.
   (a) A small category with at most one morphism between any two objects is the same thing as a *preordered set,* namely, a set with a transitive binary operation.
   (b) For any small category $\mathcal{C}$ we may define a new category $\mathcal{C}_1$ with the same objects and where there is a unique homomorphism $x \to y$ in $\mathcal{C}_1$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x,y) \neq \emptyset$. Then $\mathcal{C}_1$ is a preordered set, and there is a functor $F_1 : \mathcal{C} \to \mathcal{C}_1$ with the property that whenever $G : \mathcal{C} \to \mathcal{D}$ is a functor where $\mathcal{D}$ is a preordered set, then G can be factored $G = H \circ F_1$ for some unique functor $H : \mathcal{C}_1 \to \mathcal{D}$.

1. For any small category $\mathcal{C}$, show that the following is an equivalence relation on the objects: $x \sim y \Leftrightarrow$ there are morphisms $x \to y$ and $y \to x$. Writing $\underline{x}$ for the equivalence class of x, show that we may define a category $\mathcal{C}_2$ with these equivalence classes as objects, and where there is a morphism unique morphism $\underline{x} \to \underline{y}$ in $\mathcal{C}_2$ if and only if there is a morphism $x \to y$ in $\mathcal{C}$. Show that $\mathcal{C}_2$ is a poset, and that there is a functor $F_2 : \mathcal{C} \to \mathcal{C}_2$ with the property that whenever $G_2 : \mathcal{C} \to \mathcal{P}$ is a functor where $\mathcal{P}$ is a poset then G_2 can be factored $G_2 = H \circ F_2$ for some unique functor $H : \mathcal{C}_2 \to \mathcal{P}$.

2. Let $\mathcal{C}$ be a category which is a preordered set. Show that $\mathcal{C}$ is equivalent to a poset.

3. (A more specific version of 2.27) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be the matrix of $S : V \to V$ and let $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ be the matrix of $T : W \to W$, where V is a vector space with basis $\{v_1, v_2\}$ and W is a vector space with basis $\{w_1, w_2\}$. Assuming without proof that $\{v_i \otimes w_j \mid i,j \in \{1,2\}\}$ is a basis for $V \otimes W$, put this set in the correct order for the matrix of $\S \otimes T$ to be

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 2 & 2 & 4 \\ 3 & 0 & 4 & 0 \\ 3 & 6 & 4 & 8 \end{pmatrix}$$

Proof.

(i) p is surjective.

Let $M = B''/\operatorname{im} p$ and let $f: B'' \to B''/\operatorname{im} p$ be the natural map, so that $f \in \operatorname{Hom}(B'', M)$. Then $p^*(f) = fp = 0$, so that $f = 0$, because p^* is injective. Therefore, $B''/\operatorname{im} p = 0$, and p is surjective.

(ii) $\operatorname{im} i \subseteq \ker p$.

Since $i^* p^* = 0$, we have $0 = (pi)^*$. Hence, if $M = B''$ and $g = 1_{B''}$, so that $g \in \operatorname{Hom}(B'', M)$, then $0 = (pi)^* g = gpi = pi$, and so $\operatorname{im} i \subseteq \ker p$.

(iii) $\ker p \subseteq \operatorname{im} i$.

Now choose $M = B/\operatorname{im} i$ and let $h: B \to M$ be the natural map, so that $h \in \operatorname{Hom}(B, M)$. Clearly, $i^* h = hi = 0$, so that exactness of the Hom sequence gives an element $h' \in \operatorname{Hom}_R(B'', M)$ with $p^*(h') = h'p = h$. We have $\operatorname{im} i \subseteq \ker p$, by part (ii); hence, if $\operatorname{im} i \neq \ker p$, there is an element $b \in B$ with $b \notin \operatorname{im} i$ and $b \in \ker p$. Thus, $hb \neq 0$ and $pb = 0$, which gives the contradiction $hb = h'pb = 0$. •

The single condition that $i^*: \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(B', M)$ be surjective is much stronger than the hypotheses of Proposition 2.42 (see Exercise 2.20 on page 68).

Exercises

Unless we say otherwise, all modules in these exercises are left R-modules.

2.1 Let R and S be rings, and let $\varphi: R \to S$ be a ring homomorphism. If M is a left S-module, prove that M is also a left R-module if we define

$$rm = \varphi(r)m,$$

for all $r \in R$ and $m \in M$.

2.2 Give an example of a left R-module $M = S \oplus T$ having a submodule N such that $N \neq (N \cap S) \oplus (N \cap T)$.

***2.3** Let $f, g: M \to N$ be R-maps between left R-modules. If $M = \langle X \rangle$ and $f|X = g|X$, prove that $f = g$.

***2.4** Let $(M_i)_{i \in I}$ be a (possibly infinite) family of left R-modules and, for each i, let N_i be a submodule of M_i. Prove that

$$\Big(\bigoplus_i M_i\Big)/\Big(\bigoplus_i N_i\Big) \cong \bigoplus_i (M_i/N_i).$$

***2.5** Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left R-modules. If M is any left R-module, prove that there are exact sequences

$$0 \to A \oplus M \to B \oplus M \to C \to 0$$

and

$$0 \to A \to B \oplus M \to C \oplus M \to 0.$$

***2.6** **(i)** Let $\to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to$ be an exact sequence, and let $\operatorname{im} d_{n+1} = K_n = \ker d_n$ for all n. Prove that

$$0 \to K_n \xrightarrow{i_n} A_n \xrightarrow{d'_n} K_{n-1} \to 0$$

is an exact sequence for all n, where i_n is the inclusion and d'_n is obtained from d_n by changing its target. We say that the original sequence has been ***factored*** into these short exact sequences.

(ii) Let

$$\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} K \to 0$$

and

$$0 \to K \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \to$$

be exact sequences. Prove that

$$\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{g_0 f_0} B_0 \xrightarrow{g_1} B_1 \to$$

is an exact sequence. We say that the original two sequences have been ***spliced*** to form the new exact sequence.

***2.7** Use left exactness of Hom to prove that if G is an abelian group, then $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{I}_n, G) \cong G[n]$, where $G[n] = \{g \in G : ng = 0\}$.

***2.8** **(i)** Prove that a short exact sequence in ${}_R\mathbf{Mod}$,

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

splits if and only if there exists $q\colon B \to A$ with $qi = 1_A$. (Note that q is a retraction $B \to \operatorname{im} i$.)

(ii) A sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in **Groups** is ***exact*** if $\operatorname{im} i = \ker p$; an exact sequence

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

in **Groups** is ***split*** if there is a homomorphism $j\colon C \to B$ with $pj = 1_C$. Prove that $1 \to A_3 \to S_3 \to \mathbb{I}_2 \to 1$ is a split exact sequence. In contrast to part (i), show, in a split exact sequence in **Groups**, that there may not be a homomorphism $q\colon B \to A$ with $qi = 1_A$.

2.9** **(i)** Let $v_1, \ldots, v_n$ be a basis of a vector space V over a field k. Let $v_i^* \colon V \to k$ be the evaluation $V^* \to k$ defined by $v_i^* = (\Box, v_i)$ (see Example 1.16). Prove that $v_1^*, \ldots, v_n^*$ is a basis of V^* (it is called the ***dual basis of $v_1, \ldots, v_n$).

Hint. Use Corollary 2.22(ii) and Example 2.27.

(ii) Let $f \colon V \to V$ be a linear transformation, and let A be the matrix of f with respect to a basis $v_1, \ldots, v_n$ of V; that is, the ith column of A consists of the coordinates of $f(v_i)$ with respect to the given basis $v_1, \ldots, v_n$. Prove that the matrix of the induced map $f^* \colon V^* \to V^*$ with respect to the dual basis is the transpose A^T of A.

***2.10** If X is a subset of a left R-module M, prove that $\langle X \rangle$, the submodule of M generated by X, is equal to $\bigcap S$, where the intersection ranges over all those submodules S of M that contain X.

***2.11** Prove that if $f \colon M \to N$ is an R-map and K is a submodule of a left R-module M with $K \subseteq \ker f$, then f induces an R-map $\widehat{f} \colon M/K \to N$ by $\widehat{f} \colon m + K \mapsto f(m)$.

***2.12** **(i)** Let R be a commutative ring and let J be an ideal in R. Recall Example 2.8(iv): if M is an R-module, then JM is a submodule of M. Prove that M/JM is an R/J-module if we define scalar multiplication:

$$(r + J)(m + JM) = rm + JM.$$

Conclude that if $JM = \{0\}$, then M itself is an R/J-module. In particular, if J is a maximal ideal in R and $JM = \{0\}$, then M is a vector space over R/J.

(ii) Let I be a maximal ideal in a commutative ring R. If X is a basis of a free R-module F, prove that F/IF is a vector space over R/I and that $\{\text{cosets } x + IF : x \in X\}$ is a basis.

***2.13** Let M be a left R-module.

(i) Prove that the map $\varphi_M \colon \mathrm{Hom}_R(R, M) \to M$, given by $\varphi_M \colon f \mapsto f(1)$, is an R-isomorphism.

Hint. Make the abelian group $\mathrm{Hom}_R(R, M)$ into a left R-module by defining rf (for $f \colon R \to M$ and $r \in R$) by $rf \colon s \mapsto f(sr)$ for all $s \in R$.

(ii) If $g \colon M \to N$, prove that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_R(R, M) & \xrightarrow{\varphi_M} & M \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Hom}_R(R, N) & \xrightarrow[\varphi_N]{} & N. \end{array}$$

Conclude that $\varphi = (\varphi_M)_{M \in \mathrm{obj}({}_R\mathbf{Mod})}$ is a natural isomorphism from $\mathrm{Hom}_R(R, \Box)$ to the identity functor on ${}_R\mathbf{Mod}$. [Compare with Example 1.16(ii).]

2.14 Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of module maps. Prove that $gf = 0$ if and only if $\mathrm{im}\, f \subseteq \ker g$. Give an example of such a sequence that is not exact.

***2.15** **(i)** Prove that $f : M \to N$ is surjective if and only if $\mathrm{coker} f = \{0\}$.

(ii) If $f : M \to N$ is a map, prove that there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \mathrm{coker} f \to 0.$$

***2.16** **(i)** If $0 \to M \to 0$ is an exact sequence, prove that $M = \{0\}$.

(ii) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective if and only if h is injective.

(iii) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ be exact. If α and δ are isomorphisms, prove that $C = \{0\}$.

***2.17** If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that there is an exact sequence

$$0 \to \mathrm{coker}\, f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \to 0,$$

where $\alpha : b + \mathrm{im}\, f \mapsto gb$ and $\beta : c \mapsto hc$.

***2.18** Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence.

(i) Assume that $A = \langle X \rangle$ and $C = \langle Y \rangle$. For each $y \in Y$, choose $y' \in B$ with $p(y') = y$. Prove that

$$B = \big\langle i(X) \cup \{y' : y \in Y\} \big\rangle.$$

(ii) Prove that if both A and C are finitely generated, then B is finitely generated. More precisely, prove that if A can be generated by m elements and C can be generated by n elements, then B can be generated by $m + n$ elements.

***2.19** Let R be a ring, let A and B be left R-modules, and let $r \in Z(R)$.

(i) If $\mu_r : B \to B$ is multiplication by r, prove that the induced map $(\mu_r)_* : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B)$ is also multiplication by r.

(ii) If $m_r : A \to A$ is multiplication by r, prove that the induced map $(m_r)^* : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B)$ is also multiplication by r.

***2.20** Suppose one assumes, in the hypothesis of Proposition 2.42, that the induced map $i^*\colon \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(B', M)$ is surjective for every M. Prove that $0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$ is a split short exact sequence.

***2.21** If $T\colon \mathbf{Ab} \to \mathbf{Ab}$ is an additive functor, prove, for every abelian group G, that the function $\operatorname{End}(G) \to \operatorname{End}(TG)$, given by $f \mapsto Tf$, is a ring homomorphism.

***2.22** **(i)** Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, C) = \{0\}$ for every cyclic group C.

(ii) Let R be a commutative ring. If M is an R-module such that $\operatorname{Hom}_R(M, R/I) = \{0\}$ for every nonzero ideal I, prove that $\operatorname{im} f \subseteq \bigcap I$ for every R-map $f\colon M \to R$, where the intersection is over all nonzero ideals I in R.

(iii) Let R be a domain and suppose that M is an R-module with $\operatorname{Hom}_R(M, R/I) = \{0\}$ for all nonzero ideals I in R. Prove that $\operatorname{Hom}_R(M, R) = \{0\}$.

Hint. Every $r \in \bigcap_{I \neq 0} I$ is nilpotent.

2.23 Generalize Proposition 2.26. Let $(S_i)_{i \in I}$ be a family of submodules of a left R-module M. If $M = \langle \bigcup_{i \in I} S_i \rangle$, then the following conditions are equivalent.

(i) $M = \bigoplus_{i \in I} S_i$.

(ii) Every $a \in M$ has a unique expression of the form $a = s_{i_1} + \cdots + s_{i_n}$, where $s_{i_j} \in S_{i_j}$.

(iii) $S_i \cap \langle \bigcup_{j \neq i} S_j \rangle = \{0\}$ for each $i \in I$.

***2.24** **(i)** Prove that any family of R-maps $(f_j\colon U_j \to V_j)_{j \in J}$ can be assembled into an R-map $\varphi\colon \bigoplus_j U_j \to \bigoplus_j V_j$, namely, $\varphi\colon (u_j) \mapsto (f_j(u_j))$.

(ii) Prove that φ is an injection if and only if each f_j is an injection.

***2.25** **(i)** If $Z_i \cong \mathbb{Z}$ for all i, prove that

$$\operatorname{Hom}_{\mathbb{Z}}\Big(\prod_{i=1}^{\infty} Z_i, \mathbb{Z}\Big) \ncong \prod_{i=1}^{\infty} \operatorname{Hom}_{\mathbb{Z}}(Z_i, \mathbb{Z}).$$

Hint. A theorem of J. Łos and, independently, of E. C. Zeeman (see Fuchs, *Infinite Abelian Groups* II, Section 94) says that

$$\operatorname{Hom}_{\mathbb{Z}}\Big(\prod_{i=1}^{\infty} Z_i, \mathbb{Z}\Big) \cong \bigoplus_{i=1}^{\infty} \operatorname{Hom}_{\mathbb{Z}}(Z_i, \mathbb{Z}) \cong \bigoplus_{i=1}^{\infty} Z_i.$$

(ii) Let p be a prime and let B_n be a cyclic group of order p^n, where n is a positive integer. If $A = \bigoplus_{n=1}^{\infty} B_n$, prove that

$$\mathrm{Hom}_k\Big(A, \bigoplus_{n=1}^{\infty} B_n\Big) \ncong \bigoplus_{n=1}^{\infty} \mathrm{Hom}_k(A, B_n).$$

Hint. Prove that $\mathrm{Hom}(A, A)$ has an element of infinite order, while every element in $\bigoplus_{n=1}^{\infty} \mathrm{Hom}_k(A, B_n)$ has finite order.

(iii) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\prod_{n \geq 2} \mathbb{I}_n, \mathbb{Q}) \ncong \prod_{n \geq 2} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{I}_n, \mathbb{Q})$.

***2.26** Let R be a ring with IBN.

(i) If R^∞ is a free left R-module having an infinite basis, prove that $R \oplus R^\infty \cong R^\infty$.

(ii) Prove that $R^\infty \ncong R^n$ for any $n \in \mathbb{N}$.

(iii) If X is a set, denote the free left R-module $\bigoplus_{x \in X} Rx$ by $R^{(X)}$. Let X and Y be sets, and let $R^{(X)} \cong R^{(Y)}$. If X is infinite, prove that Y is infinite and that $|X| = |Y|$; that is, X and Y have the same cardinal.

Hint. Since X is a basis of $R^{(X)}$, each $u \in R^{(X)}$ has a unique expression $u = \sum_{x \in X} r_x x$; define

$$\mathrm{Supp}(u) = \{x \in X : r_x \neq 0\}.$$

Given a basis B of $R^{(X)}$ and a finite subset $W \subseteq X$, prove that there are only finitely many elements $b \in B$ with $\mathrm{Supp}(b) \subseteq W$. Conclude that $|B| = \mathrm{Fin}(X)$, where $\mathrm{Fin}(X)$ is the family of all the finite subsets of X. Finally, using the fact that $|\mathrm{Fin}(X)| = |X|$ when X is infinite, conclude that $R^{(X)} \cong R^{(Y)}$ implies $|X| = |Y|$.

2.2 Tensor Products

One of the most compelling reasons to introduce tensor products comes from Algebraic Topology. The homology groups of a space are interesting (for example, computing the homology groups of spheres enables us to prove the Jordan Curve Theorem), and the homology groups of the cartesian product $X \times Y$ of two topological spaces are computed (by the *Künneth formula*) in terms of the tensor product of the homology groups of the factors X and Y.

Here is a second important use of tensor products. We saw, in Example 2.2, that if k is a field, then every k-representation $\varphi\colon H \to \mathrm{Mat}_n(k)$ of a group H to $n \times n$ matrices makes the vector space k^n into a left kH-module;

Exercises

2.27 Let V and W be finite-dimensional vector spaces over a field F, say, and let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be bases of V and W, respectively. Let $S\colon V \to V$ be a linear transformation having matrix $A = [a_{ij}]$, and let $T\colon W \to W$ be a linear transformation having matrix $B = [b_{k\ell}]$. Show that the matrix of $S \otimes T\colon V \otimes_k W \to V \otimes_k W$, with respect to a suitable listing of the vectors $v_i \otimes w_j$, is the $nm \times nm$ matrix K, which we write in block form:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}.$$

Remark. The matrix $A \otimes B$ is called the ***Kronecker product*** of the matrices A and B. ◀

2.28 Let R be a domain with $Q = \mathrm{Frac}(R)$, its field of fractions. If A is an R-module, prove that every element in $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$ (instead of $\sum_i q_i \otimes a_i$). (Compare this result with Example 2.67.)

***2.29** **(i)** Let p be a prime, and let p, q be relatively prime. Prove that if A is a p-primary group and $a \in A$, then there exists $x \in A$ with $qx = a$.

(ii) If D is a finite cyclic group of order m, prove that D/nD is a cyclic group of order $d = (m, n)$.

(iii) Let m and n be positive integers, and let $d = (m, n)$. Prove that there is an isomorphism of abelian groups

$$\mathbb{I}_m \otimes \mathbb{I}_n \cong \mathbb{I}_d.$$

(iv) Let G and H be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \quad \text{and} \quad H = B_1 \oplus \cdots \oplus B_m,$$

where A_i and B_j are cyclic groups. Compute $G \otimes_{\mathbb{Z}} H$ explicitly.

Hint. $G \otimes_{\mathbb{Z}} H \cong \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j$. If A_i or B_j is infinite cyclic, use Proposition 2.58; if both are finite, use part (ii).

***2.30** **(i)** Given A_R, ${}_RB_S$, and ${}_SC$, define $T(A, B, C) = F/N$, where F is the free abelian group on all ordered triples $(a, b, c) \in A \times B \times C$, and N is the subgroup generated by all

$$(ar, b, c) - (a, rb, c),$$

$$(a, bs, c) - (a, b, sc),$$

$$(a + a', b, c) - (a, b, c) - (a', b, c),$$

$$(a, b + b', c) - (a, b, c) - (a, b', c),$$

$$(a, b, c + c') - (a, b, c) - (a, b, c').$$

Define $h\colon A \times B \times C \to T(A, B, C)$ by $h\colon (a, b, c) \mapsto a \otimes b \otimes c$, where $a \otimes b \otimes c = (a, b, c) + N$. Prove that this construction gives a solution to the universal mapping problem for triadditive functions.

(ii) Let R be a commutative ring and let $A_1, \ldots, A_n$, M be R-modules, where $n \geq 2$. An ***R-multilinear function*** is a function $h\colon A_1 \times \cdots \times A_n \to M$ if h is additive in each variable (when we fix the other $n - 1$ variables), and $f(a_1, \ldots, ra_i, \ldots, a_n) = rf(a_1, \ldots, a_i, \ldots, a_n)$ for all i and all $r \in R$. Let F be the free R-module with basis $A_1 \times \cdots \times A_n$, and define $N \subseteq F$ to be the submodule generated by all the elements of the form

$$(a_1, \ldots, ra_i, \ldots, a_n) - r(a_1, \ldots, a_i, \ldots, a_n)$$

and

$$(\ldots, a_i + a_i', \ldots) - (\ldots, a_i, \ldots) - (\ldots, a_i', \ldots).$$

Define $T(A_1, \ldots, A_n) = F/N$ and $h\colon A_1 \times \cdots \times A_n \to T(A_1, \ldots, A_n)$ by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n) + N$. Prove that h is R-multilinear, and that h and $T(A_1, \ldots, A_n)$ solve the univeral mapping problem for R-multilinear functions.

(iii) Let R be a commutative ring and prove generalized associativity for tensor products of R-modules.

Hint. Prove that any association of $A_1 \otimes \cdots \otimes A_n$ is also a solution to the universal mapping problem.

***2.31** Assume that the following diagram commutes, and that the vertical arrows are isomorphisms.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}$$

Prove that the bottom row is exact if and only if the top row is exact.

2.32** **(3×3 *Lemma) Consider the following commutative diagram in ${}_R\mathbf{Mod}$ having exact columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

***2.33** Consider the following commutative diagram in ${}_R\mathbf{Mod}$ having exact rows and columns.

$$\begin{array}{ccccccc}
A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

If $A'' \to B''$ and $B' \to B$ are injections, prove that $C' \to C$ is an injection. Similarly, if $C' \to C$ and $A \to B$ are injections, then $A'' \to B''$ is an injection. Conclude that if the last column and the second row are short exact sequences, then the third row is a short exact sequence and, similarly, if the bottom row and the second column are short exact sequences, then the third column is a short exact sequence.

2.34 Give an example of a commutative diagram with exact rows and vertical maps h_1, h_2, h_4, h_5 isomorphisms

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

for which there does not exist a map $h_3 \colon A_3 \to B_3$ making the diagram commute.

2.35** If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are categories, then a ***bifunctor $T : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ assigns, to each ordered pair of objects (A, B), where $A \in \text{ob}(\mathcal{A})$ and $B \in \text{ob}(\mathcal{B})$, an object $T(A, B) \in \text{ob}(\mathcal{C})$, and to each ordered pair of morphisms $f: A \to A'$ in $\mathcal{A}$ and $g: B \to B'$ in $\mathcal{B}$, a morphism $T(f, g): T(A, B) \to T(A', B')$, such that

(a) fixing either variable is a functor; for example, if $A \in \text{ob}(\mathcal{A})$, then $T_A = T(A, \Box): \mathcal{B} \to \mathcal{C}$ is a functor, where $T_A(B) = T(A, B)$ and $T_A(g) = T(1_A, g)$,

(b) the following diagram commutes:

$$\begin{array}{ccc} T(A, B) & \xrightarrow{T(1_A, g)} & T(A, B') \\ {\scriptstyle T(f, 1_B)}\downarrow & \searrow^{T(f,g)} & \downarrow{\scriptstyle T(f, 1_{B'})} \\ T(A', B) & \xrightarrow[T(1_{A'}, g)]{} & T(A', B'). \end{array}$$

(i) Prove that $\otimes: \mathbf{Mod}_R \times {}_R\mathbf{Mod} \to \mathbf{Ab}$ is a bifunctor.

(ii) Prove that Hom: ${}_R\mathbf{Mod} \times {}_R\mathbf{Mod} \to \mathbf{Ab}$ is a bifunctor if we modify the definition of bifunctor to allow contravariance in one variable.

***2.36** Let R be a commutative ring, and let F be a free R-module.

(i) If $\mathfrak{m}$ is a maximal ideal in R, prove that $(R/\mathfrak{m}) \otimes_R F$ and $F/\mathfrak{m}F$ are isomorphic as vector spaces over $R/\mathfrak{m}$.

(ii) Prove that $\text{rank}(F) = \dim\big((R/\mathfrak{m}) \otimes_R F\big)$.

(iii) If R is a domain with fraction field Q, prove that $\text{rank}(F) = \dim(Q \otimes_R F)$.

***2.37** Assume that a ring R has IBN; that is, if $R^m \cong R^n$ as left R-modules, then $m = n$. Prove that if $R^m \cong R^n$ as right R-modules, then $m = n$.

Hint. If $R^m \cong R^n$ as right R-modules, apply $\text{Hom}_R(\Box, R)$, using Proposition 2.54(iii).

***2.38** Let R be a domain and let A be an R-module.

(i) Prove that if the multiplication $\mu_r: A \to A$ is an injection for all $r \neq 0$, then A is ***torsion-free***; that is, there are no nonzero $a \in A$ and $r \in R$ with $ra = 0$.

(ii) Prove that if the multiplication $\mu_r: A \to A$ is a surjection for all $r \neq 0$, then A is divisible.

(iii) Prove that if the multiplication $\mu_r: A \to A$ is an isomorphism for all $r \neq 0$, then A is a vector space over Q, where $Q = \text{Frac}(R)$.

Hint. A module A is a vector space over Q if and only if it is torsion-free and divisible.

(iv) If either C or A is a vector space over Q, prove that both $C \otimes_R A$ and $\text{Hom}_R(C, A)$ are also vector spaces over Q.