Math 8211 **Homework 5** **PJW**

Date due: Monday December 3, 2012. In class on Wednesday December 5 we will grade your answers, so it is important to be present on that day, with your homework.

Rotman 7.2, 7.7 (page 417), 7.11(i), 7.14, 7.16, 7.17 (page 435), 7.20 (page 436), 7.22 (page 437) .

Questions 1 and 2 below.

1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Show that in the long exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \xrightarrow{\delta} \mathrm{Ext}^1(C, A) \to \cdots$$

   the image of 1_C under the connecting homomorphism δ is the Ext class of the extension.

2. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field k. Let us regard k as the unital R-module on which all of $x_1, \ldots, x_n$ act as 0.
   (a) Show that $\dim_k \mathrm{Ext}^1_R(k, k) = n$
   (b) Let $0 \to k^n \to E \to k \to 0$ be an extension of R-modules whose Ext class, when written in terms of components with respect to the direct sum decomposition $\mathrm{Ext}^1_R(k, k^n) \cong \bigoplus_{i=1}^n \mathrm{Ext}^1_R(k, k)$, has components which are a basis of $\mathrm{Ext}^1_R(k, k)$. Show that k^n is the unique maximal submodule of E and that E is indecomposable as an R-module (i.e. E cannot be expressed as a direct sum of two non-zero submodules). Show that E is isomorphic to $R/(x_1, \ldots, x_n)^2$.
   (c) Show that any extension of the form $0 \to k^{n+1} \to E' \to k \to 0$ must have a module E' in the middle which decomposes as an R-module.

This construction can be iterated, for $\ker D_1$ is finitely generated, and the proof is completed by induction. (We remark that we have, in fact, constructed a *free* resolution of A, each of whose terms is finitely generated.) •

Theorem 7.20. *If R is a commutative noetherian ring, and if A and B are finitely generated R-modules, then $\mathrm{Tor}_n^R(A, B)$ is a finitely generated R-module for all $n \geq 0$.*

Remark. There is an analogous result for Ext (see Theorem 7.36). ◀

Proof. Note that Tor is an R-module because R is commutative. We prove that Tor_n is finitely generated by induction on $n \geq 0$. The base step holds, for $A \otimes_R B$ is finitely generated, by Exercise 3.13 on page 115(i). If $n \geq 0$, choose a projective resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \to A \to 0$ as in Lemma 7.19. Since $P_n \otimes_R B$ is finitely generated, so are $\ker(d_n \otimes 1_B)$ (by Proposition 3.18) and its quotient $\mathrm{Tor}_n^R(A, B)$. •

Exercises

***7.1** If R is right hereditary, prove that $\mathrm{Tor}_j^R(A, B) = \{0\}$ for all $j \geq 2$ and for all right R-modules A and B.
Hint. Every submodule of a projective module is projective.

7.2 If $0 \to A \to B \to C \to 0$ is an exact sequence of right R-modules with both A and C flat, prove that B is flat.

***7.3** If F is flat and $\pi : P \to F$ is a surjection with P flat, prove that $\ker \pi$ is flat.

7.4 If A, B are finite abelian groups, prove that $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \cong A \otimes_{\mathbb{Z}} B$.

7.5 Let R be a domain with $\mathrm{Frac}(R) = Q$ and $K = Q/R$. Prove that the right derived functors of t (the torsion submodule functor) are

$$R^0 t = t, \quad R^1 t = K \otimes_R \square, \quad R^n t = 0 \quad \text{for all } n \geq 2.$$

7.6 Let k be a field, let $R = k[x, y]$, and let I be the ideal (x, y).

(i) Prove that $x \otimes y - y \otimes x \in I \otimes_R I$ is nonzero.
Hint. Consider $(I/I^2) \otimes (I/I^2)$.

(ii) Prove that $x(x \otimes y - y \otimes x) = 0$, and conclude that $I \otimes_R I$ is not torsion-free.

7.7 Prove that the functor $T = \mathrm{Tor}_1^{\mathbb{Z}}(G, \square)$ is left exact for every abelian group G, and compute its right derived functors $L_n T$.

Exercises

***7.8** **(i)** Let G be a p-primary abelian group, where p is prime. If $(m, p) = 1$, prove that $x \mapsto mx$ is an automorphism of G.

(ii) If p is an odd prime and $G = \langle g \rangle$ is a cyclic group of order p^2, prove that $\varphi\colon x \mapsto 2x$ is the unique automorphism with $\varphi(pg) = 2pg$.

***7.9** Prove that any two split extensions of modules A by C are equivalent.

7.10 Prove that if A is an abelian group with $nA = A$ for some positive integer n, then every extension $0 \to A \to E \to \mathbb{I}_n \to 0$ splits.

***7.11** **(i)** Find an abelian group B for which $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, B) \neq \{0\}$.

(ii) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, B) \neq \{0\}$ for the group B in (i).

(iii) Prove that Proposition 7.39 may be false when A is not finitely generated, even when $R = \mathbb{Z}$.

***7.12** Let E be a left R-module. Prove that E is injective if and only if $\text{Ext}^1_R(A, E) = \{0\}$ for every left R-module A.

***7.13** **(i)** Prove that the covariant functor $E = \text{Ext}^1_{\mathbb{Z}}(G, \Box)$ is right exact for every abelian group G, and compute its left derived functors $L_n E$.

(ii) Prove that the contravariant functor $F = \text{Ext}^1_{\mathbb{Z}}(\Box, G)$ is right exact for every abelian group G, and compute its left derived functors $L_n F$. (See the footnote on page 370.)

7.14 **(i)** If A is an abelian group with $mA = A$ for some nonzero $m \in \mathbb{Z}$, prove that every exact sequence $0 \to A \to G \to \mathbb{I}_m \to 0$ splits. Conclude that $m\,\text{Ext}^1_{\mathbb{Z}}(A, B) = \{0\} = m\,\text{Ext}^1_{\mathbb{Z}}(B, A)$.

(ii) If A and C are abelian groups with $mA = \{0\} = nC$, where $(m, n) = 1$, prove that every extension of A by C splits.

7.15 **(i)** For any ring R, prove that a left R-module B is injective if and only if $\text{Ext}^1_R(R/I, B) = \{0\}$ for every left ideal I.

Hint. Use the Baer criterion.

(ii) If D is an abelian group and $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = \{0\}$, prove that D is divisible. The converse is true because divisible abelian groups are injective. Does this hold if we replace $\mathbb{Z}$ by a domain R and $\mathbb{Q}/\mathbb{Z}$ by $\text{Frac}(R)/R$?

7.16 Let G be an abelian group G. Prove that G is free abelian if and only if $\text{Ext}^1_{\mathbb{Z}}(G, F) = \{0\}$ for every free abelian group F.

***7.17** Let A be a torsion abelian group and let S^1 be the circle group. Prove that $\text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(A, S^1)$.

7.18** An abelian group W is a ***Whitehead group if $\mathrm{Ext}^1_{\mathbb{Z}}(W, \mathbb{Z}) = \{0\}$.[3]

(i) Prove that every subgroup of a Whitehead group is a Whitehead group.

(ii) Prove that $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, S^1)$ if A is a torsion group and S^1 is the circle group. Prove that if $A \neq \{0\}$ is torsion, then A is not a Whitehead group; conclude further that every Whitehead group is torsion-free.

Hint. Use Exercise 7.17.

(iii) Let A be a torsion-free abelian group of rank 1; i.e., A is a subgroup of $\mathbb{Q}$. Prove that $A \cong \mathbb{Z}$ if and only if $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \neq \{0\}$.

(iv) Let A be a torsion-free abelian group of rank 1. Prove that if A is a Whitehead group, then $A \cong \mathbb{Z}$.

Hint. Use an exact sequence $0 \to \mathbb{Z} \to A \to T \to 0$, where T is a torsion group whose p-primary component is either cyclic or isomorphic to Prüfer's group of type p^∞.

(v) (**K. Stein**). Prove that every countable[4] Whitehead group is free abelian.

Hint. Use Exercise 3.4 on page 114, *Pontrjagin's Lemma*: if A is a countable torsion-free group and every subgroup of A having finite rank is free abelian, then A is free abelian.

7.19 We have constructed the bijection $\psi\colon e(C, A) \to \mathrm{Ext}^1(C, A)$ using a projective resolution of C. Define a function $\psi'\colon e(C, A) \to \mathrm{Ext}^1(C, A)$ using an injective resolution of A, and prove that ψ' is a bijection.

7.20 Consider the diagram

$$\begin{array}{lccccccccc}
\xi_1 = & 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & & {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle k} & & \\
\xi_2 = & 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0.
\end{array}$$

Prove that there is a map $\beta\colon B_1 \to B_2$ making the diagram commute if and only if $[h\xi_1] = [\xi_2 k]$.

7.21 **(i)** Prove, in $e(C, A)$, that $-[\xi] = [(-1_A)\xi] = [\xi(-1_C)]$.

(ii) Generalize (i) by replacing (-1_A) and (-1_C) by μ_r for any r in the center of R.

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[3] Dixmier proved that a locally compact abelian group A is path connected if and only if $A \cong \mathbb{R}^n \oplus \widehat{D}$, where D is a (discrete) Whitehead group and $\widehat{D}$ is its Pontrjagin dual.

[4] The question whether $\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = \{0\}$ implies G is free abelian is known as ***Whitehead's problem***. S. Shelah proved that it is undecidable whether uncountable Whitehead groups must be free abelian (see Eklof, "Whitehead's problem is undecidable," *Amer. Math. Monthly* 83 (1976), 775–788).

7.22 Prove that $[\xi] = [0 \to A \xrightarrow{i} B \to C \to 0] \in e(C, A)$ has finite order if and only if there are a nonzero $m \in \mathbb{Z}$ and a map $s\colon B \to A$ with $si = m \cdot 1_A$.

***7.23** **(i)** Prove that $e(C, \Box)\colon {}_R\mathbf{Mod} \to \mathbf{Ab}$ is a covariant functor if, for $h\colon A \to A'$, we define $h_*\colon e(C, A) \to e(C, A')$ by $[\xi] \mapsto [h\xi]$.

(ii) Prove that $e(C, \Box)$ is naturally isomorphic to $\mathrm{Ext}^1_R(C, \Box)$.

7.24 Consider the extension $\chi = 0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$.

(i) Define $D\colon \mathrm{Hom}_R(C, A'') \to e(C, A')$ by $k \mapsto [\chi k]$, and prove exactness of

$$\mathrm{Hom}(C, A) \xrightarrow{p_*} \mathrm{Hom}(C, A'') \xrightarrow{D} e(C, A') \xrightarrow{i_*} e(C, A) \xrightarrow{p_*} e(C, A'').$$

(ii) Prove commutativity of

$$\begin{array}{ccc} \mathrm{Hom}(C, A'') & \xrightarrow{D} & e(C, A') \\ & \searrow^{\partial} & \downarrow{\scriptstyle \psi} \\ & & \mathrm{Ext}^1(C, A'), \end{array}$$

where ∂ is the connecting homomorphism.

7.25 **(i)** Prove that $e(\Box, A)\colon {}_R\mathbf{Mod} \to \mathbf{Ab}$ is a contravariant functor if, for $k\colon C' \to C$, we define $k^*\colon e(C, A) \to e(C', A)$ by $[\xi] \mapsto [\xi k]$.

(ii) Prove that $e(\Box, A)$ is naturally isomorphic to $\mathrm{Ext}^1_R(\Box, A)$.

***7.26** Consider the extension $X = 0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$.

(i) Define $D'\colon \mathrm{Hom}_R(C', A) \to e(C'', A)$ by $h \mapsto [hX]$, and prove exactness of

$$\mathrm{Hom}(C, A) \xrightarrow{i^*} \mathrm{Hom}(C', A) \xrightarrow{D'} e(C'', A) \xrightarrow{p^*} e(C, A) \xrightarrow{i^*} e(C', A).$$

(ii) Prove commutativity of

$$\begin{array}{ccc} \mathrm{Hom}(C', A) & \xrightarrow{D'} & e(C'', A) \\ & \searrow^{\partial} & \downarrow{\scriptstyle \psi} \\ & & \mathrm{Ext}^1(C'', A), \end{array}$$

where ∂' is the connecting homomorphism.