Worksheet for Matsumura Chapter 2: Prime ideals

1. Why is the following true?:

PROPOSITION. *Suppose that $f : A \to B$ is a ring homomorphism satisfying the two conditions*

(1) $f(x)$ is a unit of B for all $x \in S$;

(2) if $g : A \to C$ is a homomorphism of rings taking every element of S to a unit of C then there exists a unique homomorphism

$$h : B \to C \qquad \textit{such that} \qquad g = hf;$$

then B is uniquely determined up to isomorphism.

2. Why does Matsumura define the localisation in this way? Why not just construct it, like the way we introduce fractions at school?
3. Check for yourself that the relation $\sim$ on $A \times S$ introduced at the bottom of page 20 is an equivalence relation. True or false: if A is an integral domain and S is a multiplicative subset then

$$(a, s) \sim (b, s') \Leftrightarrow s'a = sb$$

is an equivalence relation.

4. Is the following result easy to prove or difficult to prove?

PROPOSITION. *Let $f : A \to A_S$ be the map $f(a) = a/1$. Then*

$$\operatorname{Ker} f = \{a \in A \mid sa = 0 \textit{ for some } s \in S\}.$$

5. On page 21, prove that there is a bijection

$$\{IB \mid I \triangleleft A\} \leftrightarrow \{J \cap A \mid J \triangleleft B\}$$

6. If $P \triangleleft B$ is a prime ideal then $P \cap A$ is a prime ideal of A. Can you find an example of a prime ideal $P \triangleleft A$ for which PB is not a prime ideal?
7. Which is always true?: a prime ideal is always primary; a primary ideal is always prime.
8. Prove:

PROPOSITION.

(1) $J \triangleleft B$ is primary $\Leftrightarrow$ zero divisors of B/J are nilpotent.

(2) If $f : A \to B$ and J is primary in B then $J \cap A$ is primary.

9. Exercise 4.1 is: If J is primary then $\sqrt{J}$ is a prime ideal. What about the converse: If $\sqrt{J}$ is a prime ideal then J is primary?
10. Is the following true: if S is a multiplicatively closed subset of A and I is an ideal of A then every element of IA_S can be written x/s where $x \in I$.
11. On page 22 there is a statement

$$\frac{a}{s} \cdot \frac{b}{t} \in IA_S \quad \text{with} \quad s, t \in S \Rightarrow rab \in I \quad \text{for some} \quad r \in S.$$

    Prove this.
12. True or false: if $0 \in S$ then $A_S = 0$.