

Date due: November 16, 2015.

In the questions which follow about $SL(2, \mathbb{Z})$ we will denote

$$\alpha := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

- Let $F(x, y)$ be a free group on generators x and y and let $\Gamma = \Gamma(F(x, y), \{x, y\})$ be the tree which is the Cayley graph with respect to these generators. For each $n \geq 0$ let $H_n = \langle x, {}^y x, {}^{y^2} x, \dots, {}^{y^n} x \rangle$ as a subgroup of $F(x, y)$.
 - Draw a picture of the quotient $H_n \backslash \Gamma$.
 - Show that H_n is a free group of rank n .
 - Show that $H = \langle {}^{y^i} x \mid i \in \mathbb{N} \rangle$ is a free subgroup of $F(x, y)$ of infinite rank.
- Let F be free on generators x and y and let $\phi : F \rightarrow S_3$ be the homomorphism determined by $\phi(x) = (1, 2)$ and $\phi(y) = (2, 3)$. Let N be the kernel of ϕ . Find a set of free generators for N .
 [You may assume without proof that N is indeed a free group and that N is generated as a *normal* subgroup of F (not as a free group) by x^2, y^2 and $xyxyxy$.]

- Express the matrices

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

as products of the generators α and β of $SL(2, \mathbb{Z})$.

- Suppose that H and J are subgroups of G . The coset graph $C(G; H, J)$ of G with respect to H and J has the left cosets xH and yJ of H and J as its vertices, and there is an edge joining xH and yJ if and only if $xH \cap yJ \neq \emptyset$. There is a left action of G on $C(G; H, J)$ as a group of graph automorphisms, given by left multiplication.
 - Draw a picture of $C(C_{12}; C_4, C_6)$
 - Show that there is a single G -orbit of edges in $C(G; H, J)$.
 - Describe a fundamental domain for the action of G on $C(G; H, J)$, indication stabilizers of all vertices and edges in your fundamental domain.
 - Suppose that $K \triangleleft G$. Show that

$$K \backslash C(G; H, J) \cong C(G/K; HK/K, JK/K).$$

- Show that the coset graph $C(SL(2, \mathbb{Z}); \langle \alpha \rangle, \langle \beta \rangle)$ is a tree
 - Let K be kernel of the homomorphism $SL(2, \mathbb{Z}) \rightarrow C_{12} = \langle x \rangle$ which sends α to x^3 and β to x^2 . Find the rank of K as a free group. Find a set of matrices which are free generators for K . [Questions 4 and 5a are intended to help with this.]

6. (a) Let G be the group of permutations of \mathbb{Z} generated by the two mappings α and β defined by $\alpha(x) = -x$ and $\beta(x) = -x + 1$. Show that G is isomorphic to $C_2 * C_2$. Let a, b denote generators of the two copies of C_2 in this free product.
 (b) Show that $G \cong C_\infty \rtimes C_2$ where C_∞ is an infinite cyclic subgroup. Identify a generator of this subgroup as a word in a and b .
 (c) Show that every subgroup of $C_2 * C_2$ is isomorphic to $1, C_2, C_\infty$ or $C_2 * C_2$.
7. Is $C_2 * C_2$ isomorphic to a subgroup of $SL(2, \mathbb{Z})$? Is $C_2 * C_2$ isomorphic to a subgroup of $PSL(2, \mathbb{Z})$?
 [Note (and prove if you want to) that $C_2 * C_2$ is isomorphic to the group of 2 by 2 integer matrices generated by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.]

Extra questions – do not hand in:

8. Let G be the free group on generators a and b . Prove that G is not generated by $\{a, bab\}$ or by $\{aba, bab\}$. [This shows that we cannot take the generating set $\{a, aba\}$ and replace a by either element of $\{b, bab\}$ and still have a generating set, in contrast to what happens with vector spaces. Generating sets do not form a matroid.]
9. Let $G = \langle a, b \rangle$ where $a = (1, 2, 3, 4)$ and $b = (3, 4, 5, 6)$. Find a Schreier transversal in terms of a and b for $\text{Stab}_G(2)$, the stabilizer in G of the symbol 2.
10. Let F be a free group on a subset X . If $x \in X$ and $f \in F$, define $\sigma_x(f)$ to be the sum of the exponents of x in the reduced form of f . Prove that $f \in F'$ if and only if $\sigma_x(f) = 0$ for all x in X . [Here F' is the derived subgroup, which is generated by the commutators of elements in F .]
11. The Mathieu group M_{24} may be generated by permutations

$$(1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21)(22, 23)$$

and

$$(1, 2, 5, 7, 15, 20, 14, 23, 21, 11, 16, 19, 24, 6, 8, 4, 17, 3, 10, 13, 18)(9, 22, 12).$$

- a) Make a stabilizer chain for M_{24} and determine the lengths of the orbits $\Delta^{(i)}$.
 b) What is the smallest size of a base for a group of size $|M_{24}|$ acting on 24 points?

12. The Mathieu group M_{12} may be generated by permutations

$$(1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12) \quad \text{and} \quad (1, 9, 12, 7, 11)(6, 2, 8, 3, 5).$$

Same question as for M_{24} .