Math 8246 **Homework 3** **PJW**

Date due: Monday April 11, 2016. We will discuss these questions on Wednesday 4/13/2016

1. For each of the crystal structures labeled 2 and 3 on the sheet with three Escher designs (lizards and frogs, angels and bats), determine the point group, and identify the equivalent crystal structure on the list of 17 wallpaper patterns.

2. Provide a proof that there are two arithmetic crystal classes of wallpaper patterns with point group D_6. Show that the class in which D_6 acts via the matrices

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

   gives rise to the crystal structure p31m (rather than p3m1).

3. Assume that the only finite orders of orthogonal transformations of 2-dimensional space that preserve a lattice isomorphic to $\mathbb{Z}^2$ (containing two independent vectors) are 1, 2, 3, 4 and 6. Write out a proof that there is no crystal structure in dimension 3 with icosahedral symmetry. Pay careful attention to the step where you show that you have an action on a discrete 2-dimensional lattice, rather than some other group.

4. Making sure that the generators you use for the action on $\mathbb{Z}^2$ have the order you think they have, choose a presentation for D_8 and compute $H^2(D_8, T)$, where T is the only lattice $\mathbb{Z}^2$ on which D_8 acts. Identify which of the two space groups p4mm and p4gm correspond to the split extension, and compute the largest possible point stabilizer in the non-split extension, in its action on 2-dimensional space.

5. Show that the group with the Coxeter presentation

$$\langle s_1, s_2, s_3 \mid s_1^2, s_1^2, s_3^2, (s_1 s_2)^3, (s_1 s_3)^3, (s_2 s_3)^3 \rangle$$

   is a crystallographic group in dimension 2. Identify it on the list of 17 wallpaper patterns. Find generators for the translation subgroup as words in the Coxeter generators s_1, s_2, s_3.

6. Let R be a (not necessarily commutative) ring. Show that every homomorphism of free left R-modules $\phi : R^p \to R^q$ determines, and is determined by, a matrix $A_\phi = (a_{i,j})$ with the property that if $\mathbf{v} = (v_1, \dots, v_p)^T$ is a column vector $\mathbf{v} \in R^p$ then $\phi(\mathbf{v}) = A\mathbf{v}$, where $A\mathbf{v}$ has entries $\sum_j v_j a_{i,j}$. Show that if $B_\psi = (b_{k,i})$ is the matrix of another map $\psi : R^q \to R^r$ then the composite $\psi\phi$ has matrix with entries $(\sum_i a_{i,j} b_{k,i})$.

Extra: Investigate KaleidoPaint at http://www.geometrygames.org/ and the image gallery http://www.geometrygames.org/KaleidoPaint/Gallery/index.html