Free groups acting on trees

Let G act on a set Ω (from the right) and let X be a set of generators of G. The Cayley graph $\Gamma(\Omega, X)$ is the graph whose vertex set is Ω and where there is an edge $\omega_1 \to \omega_2$ labeled by $x \in X$ if and only if $\omega_2 = \omega_1 x$.

PROPOSITION. *If F is the free group generated by X then $\Gamma(F, X)$ is a tree.*

Proof. Each non-identity vertex $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ has one adjacent vertex of shorter length, namely $x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}}$, and all other adjacent vertices are strictly longer. If there were a circuit in the graph, consider a minimal circuit and a vertex in it of maximal length: it must have two adjacent vertices of shorter length. These two adjacent vertices must be equal, which contradicts minimality of the circuit. □

If $\Omega = G$ is the regular G-set then G acts on $\Gamma(G, X)$ from the left, with an element $g \in G$ sending an edge $y \to yx$ to $gy \to gyx$.

PROPOSITION. *The action of G on $\Gamma(G, X)$ is properly discontinuous: it is a simplicial action and no edge or vertex is fixed by any element of G other than the identity. Hence the quotient map $p : \Gamma(G, X) \to G\backslash\Gamma(G, X)$ is a covering. If G is freely generated by X, it is a universal covering.*

LEMMA. *If G acts properly discontinuously on a tree Γ then $\pi_1(G\backslash\Gamma, x_0) \cong G$, where x_0 is a base point.*

Proof. Let $\hat{x}_0$ be a base point in Γ with $p(\hat{x}_0) = x_0$. We obtain a mapping $G \to \pi_1(G\backslash\Gamma, x_0)$ as follows: for each $g \in G$ there is a unique (shortest) path α_g from $\hat{x}_0$ to $g\hat{x}_0$ and we send g to the homotopy class $[p(\alpha_g)]$. Given a circuit in $(G\backslash\Gamma, x_0)$ based at x_0 it has a unique lift to a path in Γ starting at $\hat{x}_0$. The end point of the path has the form $g\hat{x}_0$ for some g, and depends only on the based homotopy type of the path. We send the circuit to g, and this defines a mapping $\pi_1(G\backslash\Gamma, x_0) \to G$. These two mappings are inverse and are group homomorphisms when paths are multiplied on the right. □

PROPOSITION. *If U is any graph with distinguished vertex u_0 then*

(a) $\pi_1(U, u_0)$ *is a free group of rank* $-\tilde{\chi}(U)$.

(b) Let T be a maximal subtree of U and for each vertex v let α_v be the geodesic from u_0 to v in T. Then the elements $\alpha_v \beta \alpha_w^{-1}$ ranging over edges β from v to w which are not in T freely generate $\pi_1(U, u_0)$.

THEOREM.

(a) Let G be a group. Then G is a free group if and only if G can act properly discontinuously on a tree.

(b) Subgroups of free groups are free.

LEMMA. *If H is a subgroup of G, which has generator set X, then*

$$H\backslash\Gamma(G,X) = \Gamma(H\backslash G, X).$$

THEOREM. *If H is a subgroup of a free group F of finite index n and the rank $d(F)$ is finite then $d(H) - 1 = n(d(G) - 1)$. Furthermore, H is freely generated by the non-identity elements of the form $tx\overline{tx}^{-1}$ where t ranges over the elements of a right Schreier transversal to H in G, x ranges over the generators of G, and $\overline{y}$ is the transversal element representing the same coset as y.*

Definition: If $G = \langle x_i \mid r_j\rangle$ and $H = \langle y_k \mid s_l\rangle$ the free product of G and H is the group with presentation $G * H = \langle x_i, y_k \mid r_j, s_l\rangle$. There are group homomorphisms $\theta : G \to G * H$ and $\phi : H \to G * H$. Every element of $G * H$ can be written as a product $\theta(g_1)\phi(h_1)\cdots\theta(g_n)\phi(h_n)$ for some n, where the g_i lie in G and the h_j lie in H. If K is identified as a subgroup of G and also of H (via some injective homomorphism $\alpha : K \to H$) we define the free product with amalgamation $G *_K H = (G * H)/N$ where N is the normal subgroup generated by elements $\theta(y)\phi(\alpha(y))^{-1}$ as y ranges through elements of K.

PROPOSITION. *$SL(2,\mathbb{Z})$ acts on $\mathbb{C}$ by Moebius transformations. These send circles and straight lines to circles and straight lines, preserving orthogonality. It preserves the real axis and also the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The center of $SL(2,\mathbb{Z})$ is generated by $-I_2$ and acts trivially. Thus we obtain an action of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I_2\}$.*

PROPOSITION. *$SL(2,\mathbb{Z})$ is generated by elements*

$$\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

of orders 4 and 6 with square equal to $-I_2$. Hence we obtain homomorphisms $C_4 *_{C_2} C_6 \to SL(2,\mathbb{Z})$ and $C_2 * C_3 \to PSL(2,\mathbb{Z})$. These two homomorphisms are isomorphisms.