G-sets and Stabilizer Chains

Let G be a group. A *G-set* is a set Ω with an action of G by permutations. Distinguishing between right and left G-sets, by a *right* G set we mean that there is a mapping $\Omega \times G \to \Omega$ so that $\omega(gh) = (\omega g)h$ and $\omega \cdot 1 = \omega$ always hold. It is equivalent to require that there be a homomorphism $G \to S_\Omega$, the symmetric group on Ω (with functions applied from the right).

A *homomorphism* $f : \Omega \to \Psi$ of G-sets is a mapping with $f(\omega g) = (f(\omega))g$ always. Such a homomorphism is an isomorphism if and only if it is bijective, if and only if there is a G-set homomorphism $f_1 : \Psi \to \Omega$ with $1_\Psi = f f_1$ and $1_\Omega = f_1 f$.

If $\omega \in \Omega$ the set $\omega G = \{\omega g \mid g \in G\}$ is the orbit of Ω which contains ω. We say that G acts *transitively* on Ω if there is only one orbit. We put

$$\mathrm{Stab}_G(\omega) = G_\omega = \{g \in G \mid \omega g = \omega\}$$

and this is the *stabilizer* of ω in G. For example:

- if G permutes the set of its subgroups by conjugation then $\mathrm{Stab}_G(H) = N_G(H)$,
- if G permutes the set of its elements by conjugation then $\mathrm{Stab}_G(x) = C_G(x)$,
- if G permutes the right cosets $H\backslash G = \{Hg \mid g \in G\}$ by right multiplication then $\mathrm{Stab}_G(Hg) = H^g$.

PROPOSITION.

(1) Every G-set Ω has a unique decomposition $\Omega = \bigcup_{i \in I} \Omega_i$ where I is some indexing set and the Ω_i are orbits of Ω.

(2) If Ω is transitive and $\omega \in \Omega$ then $\Omega \cong \mathrm{Stab}_G(\omega)\backslash G$ as G-sets. Consequently, if Ω is finite then $|\Omega| = |G : \mathrm{Stab}_G(\omega)|$.

(3) If $H, K \leq G$ then $H\backslash G \cong K\backslash G$ as G-sets if and only if K and H are conjugate subgroups of G.

PROPOSITION.

(1) Every map between transitive G-sets is an epimorphism.

(2) $\mathrm{Aut}_{G-\mathrm{set}}(H\backslash G) \cong N_G(H)/H$.

(3) Every homomorphism $H\backslash G \to K\backslash G$ has the form $H\backslash G \to J\backslash G \to K\backslash G$ where $H \leq J$, $H\backslash G \to J\backslash G$ is the morphism $Hx \mapsto Jx$, and J is conjugate to K.

Let H be a subgroup of a group G. A *right transversal* to H in G is the same thing as a set of right coset representatives for H in G, that is a set of elements $g_1, \ldots, g_t$ of G so that $G = Hg_1 \cup \cdots \cup Hg_t$.

PROPOSITION. *Let G act transitively on a set Ω and let $\omega \in \Omega$ be an element with stabilizer G_ω. Then elements $\{g_i \mid i \in I\}$ of G form a right transversal to G_ω in G if and only if $\Omega = \{\omega g_i \mid i \in I\}$ and the ωg_i are all distinct.*

Proof. This comes from the isomorphism of G-sets $\Omega \cong G_\omega \backslash G$ under which $\omega g \leftrightarrow G_\omega g$. □

This observation provides a way to compute a transversal for $\mathrm{Stab}_G(\omega)$ in G. Take the generators of G and repeatedly apply them to ω, obtaining various elements of the form $\omega g_{i_1} g_{i_2} \cdots g_{i_r}$ where the g_{i_j} are generators of G. Each time we get an element we have seen previously, we discard it. Eventually we obtain the orbit ωG, and the various elements $g_{i_1} g_{i_2} \cdots g_{i_r}$ are a right transversal to $\mathrm{Stab}_G(\omega)$ in G.

This is what GAP does, except that it does the above with the inverses of the generators of G. If an inverse generator g^{-1} sends an already-computed element u to a new element v, the generator g is stored in position v in a list. This means that applying g to v gives u. By repeating this we eventually get back to the first element of the orbit. It is this list of generators that GAP stores in the field '`transversal`' of a stabilizer chain. Elements of a right transversal are obtained by multiplying the inverses of the generators in reverse sequence.

Computing chains of stabilizers is the most important technique available in computations with permutation groups. It is necessary to compute generators for stabilizer subgroups and this is done by the following theorem.

THEOREM (Schreier). *Let X be a set of generators for a group G, $H \leq G$ a subgroup, and T a right transversal for H in G such that the identity element of G represents the coset H. For each $g \in G$ let $\overline{g} \in T$ be such that $H\overline{g} = Hg$. Then*

$$\{tg(\overline{tg})^{-1} \mid t \in T, g \in X\}$$

is a set of generators for H.

Note that since $Htg = H\overline{tg}$, the elements $tg(\overline{tg})^{-1}$ lie in H always. Also $\overline{\overline{a}} = \overline{a}$ and $\overline{\overline{a}b} = \overline{ab}$. The generators in the set are called *Schreier generators*.

Proof. Suppose that $g_1 \cdots g_n \in H$ where the g_i lie in X. Then

$$g_1 \cdots g_n = (g_1 \overline{g_1}^{-1})(\overline{g_1} g_2 \overline{g_1 g_2}^{-1})(\overline{g_1 g_2} g_3 \overline{g_1 g_2 g_3}^{-1} \cdots (\overline{g_1 \cdots g_{n-1}} g_n)$$

is a product of the Schreier generators. Note that $g_1 \cdots g_n \in H$ so that $\overline{g_1 \cdots g_n} = 1$. □

If G permutes Ω, a *base* for G on Ω is a list of elements $\omega_1, \omega_2, \ldots, \omega_s$ of Ω so that the stabilizer $G_{\omega_1,\omega_2,\ldots,\omega_s}$ equals 1. Here $G_{\omega_1,\omega_2,\ldots,\omega_r}$ is the stabilizer inside the subgroup $G_{\omega_1,\omega_2,\ldots,\omega_{r-1}}$ of ω_r, for each r. Let us write G_r instead of $G_{\omega_1,\omega_2,\ldots,\omega_r}$ and $G_0 = G$. In this situation the chain of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_s = 1$$

is called a *stabilizer chain* (for G, with respect to the given base). We will consider for each r the subset Ω_r of Ω which is defined to be the G_r-orbit containing ω_{r+1}. Thus $\Omega_0 = \omega_1 G$, $\Omega_1 = \omega_2 G_1$ etc. A *strong generating set* for G (with respect to the base) is a set of generators for G which includes generators for each of the subgroups G_r. Thus in a strong generating set, G_r is generated by those generators which happen to fix each of $\omega_1, \ldots, \omega_r$.

PROPOSITION. *Each Ω_i is acted on transitively by G_i. As G_i-sets, $\Omega_i \cong G_{i+1}\backslash G_i$. Hence $|G| = |\Omega_0| \cdots |\Omega_{s-1}|$.*

Proof. We have $\omega_{i+1} \in \Omega_i$ and $\mathrm{Stab}_{G_i}(\omega_{i+1}) = G_{i+1}$. □

Given a stabilizer chain we obtain an algorithm to test whether a given permutation π of Ω is an element of G. We compute $(\omega_1)\pi$. If $\pi \in G$ this must equal $(\omega_1)g$ for some unique g in a right transversal for G_1 in G_0 and so $\pi g^{-1} \in G_1$. In fact, $\pi \in G$ if and only if $(\omega_1)\pi = (\omega_1)g$ for some g in the transversal and $\pi g^{-1} \in G_1$. We now continue to test whether $\pi g^{-1} \in G_1$ by repeating the algorithm.

Given a set of generators $G = \langle g_1, \ldots, g_d \rangle$ and a subgroup $H \leq G$ a *right Schreier transversal* for H in G is a right transversal with elements expressed as words in the generators, as suggested by the following $1, g_{i_1}, g_{i_1}g_{i_2}, g_{i_3}, \ldots$ so that each initial segment of a word appears (earlier) in the list. Schreier transversals correspond to rooted trees.

THEOREM. *(Schreier) Let G have d generators and let $H \leq G$ have finite index. Then H can be generated by e elements, where $(e-1) \leq |G : H|(d-1)$.*

Proof. Consider the generators $tg(\overline{tg})^{-1}$ for H, and write $n = |G : H|$. The number of edges in the Schreier tree is $n-1$. Each gives an entry 1 in the table of generators. The number of table entries which are not 1 is at most $dn - n + 1 = n(d-1) + 1$. □

When G is a free group the bound on $d(H)$ is always achieved.