Math 8300 **Homework 3** **PJW**

We will discuss these questions on Friday May 5, 2017

These questions can all be done using technology presented in class. It would be possible to do some of them in a different way, perhaps by studying various texts. The point about these questions is that they reinforce what is done in class, and I prefer it if you use the methods I have taught.

1. (8 points) Give a proof of the following result by following the suggested steps.

   THEOREM. *Let $E \supset F$ be a field extension of finite degree and let A be an F-algebra. Let U and V be A-modules. Then*

   $$E \otimes_F \mathrm{Hom}_A(U, V) \cong \mathrm{Hom}_{E\otimes_F A}(E \otimes_F U, E \otimes_F V)$$

   via an isomorphism $\lambda \otimes_F f \mapsto (\mu \otimes_F u \mapsto \lambda\mu \otimes_F f(u))$.

   (a) Verify that there is indeed a homomorphism as indicated.

   (b) Let $x_1, \ldots, x_n$ be a basis for E as an F-vector space. Show that for any F-vector space M, each element of $E \otimes_F M$ can be written uniquely in the form $\sum_{i=1}^n x_i \otimes_F m_i$ with $m_i \in M$.

   (c) Show that if an element $\sum_{i=1}^n x_i \otimes f_i \in E \otimes_F \mathrm{Hom}_A(U, V)$ maps to 0 then $\sum_{i=1}^n x_i \otimes f_i(u) = 0$ for all $u \in U$. Deduce that the homomorphism is injective.

   (d) Show that the homomorphism is surjective as follows: given an $E \otimes_F A$-module homomorphism $g : E \otimes_F U \to E \otimes_F V$, write $g(1 \otimes_F u) = \sum_{i=1}^n x_i \otimes f_i(u)$ for some elements $f_i(u) \in V$. Show that this defines A-module homomorphisms $f_i : U \to V$. Show that g is the image of $\sum_{i=1}^n x_i \otimes f_i$.

2. (5 points) The antiautomorphism of $S_F(n, r)$ used in defining the dual of a representation of the Schur algebra was defined as sending an endomorphism of $E^{\otimes r}$ to its transpose with respect to the standard bilinear form on $E^{\otimes r}$. Compute the effect of this antiautomorphism on the basis elements $\xi_{\mathbf{i},\mathbf{j}}$ of $S_F(n, r)$ constructed as the duals of the monomial functions $c_{\mathbf{i},\mathbf{j}}$.

3. (5 points) For any finite dimensional representation V of a group G we can construct another representation V^* whose representation space is $\mathrm{Hom}_F(V, F)$ and where $g \in G$ acts on a linear map $f : V \to F$ to give ${}^g f$, where ${}^g f(v) = f(g^{-1}v)$. Suppose that F is infinite and V is a polynomial representation of $GL_n(F)$. Show that V^* is polynomial if and only if $GL_n(F)$ acts trivially on V.

4. (5 points) Show that the simple $S_F(n, r)$-modules are self-dual.

5. (5 points) In the situation where we have an algebra B containing an idempotent e and a Schur functor $f : B\text{-mod} \to eBe\text{-mod}$, show that the left adjoint and the right adjoint functors of f need not be naturally isomorphic. The left adjoint is $W \mapsto Be \otimes_{eBe} W$ and the right adjoint is $W \mapsto \mathrm{Hom}_{eBe}(eB, W)$.