Topics in Group Representation Theory

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Contents

Landmark Theorems

0.1 Representations of S_r

Theorem 0.1.1. *The simple representations D^λ of S_r over $\mathbb{F}_p$ are parametrized by p-regular partitions λ of r. They are self-dual and absolutely irreducible.*

Other properties: The Specht module S^λ has one composition factor D^λ and all others are D^μ with $\mu \trianglerighteq \lambda$.

Theorem 0.1.2. *As μ ranges over partitions of r, the indecomposable summands Y^λ of $M^\mu = \mathbb{F}_p \uparrow_{S_\mu}^{S_r}$ are parametrized by the partitions λ of r. Each permutation module M^μ has a summand Y^μ with multiplicity 1, and all other summands M^λ have $\lambda \trianglerighteq \mu$.*

Other properties: The M^λ have a Specht filtration. M^λ is projective if and only if λ' is p-regular.

We will avoid the usual things done in characteristic zero in other courses: construction of the character table, hook length formula, Murnaghan-Nakayama etc etc.

0.2 Polynomial representations of $GL_n(k)$

Let k be a field of characteristic p and $E = k^n$. S_r acts on $E^{\otimes r}$ by permuting the positions of the tensor factors, and $E^{\otimes r}$ is a direct sum of permutation modules M^λ. We define $S_k(n,r) = \mathrm{End}_{kS_r}(E^{\otimes r})$. This is the *Schur algebra* associated to this situation.

Theorem 0.2.1. *$S_k(n,r)$ is quasi-hereditary. When $n \geq r$ all the $S_k(n,r)$ are Morita equivalent. The simple $S_k(n,r)$-modules are parametrized by partitions of r into at most n parts.*

Theorem 0.2.2. *The polynomial representations of $GL(n(k))$ are direct sums of homogeneous polynomial representations of various degrees r.*

Theorem 0.2.3. *When k is infinite the algebra homomorphism $k[GL_n(k)] \to S_k(n,r)$ is surjective. The polynomial representations of degree r are the same as the representations of $S_k(n,r)$. When $n \geq r$ the simple polynomial representations of degree r are parametrized by the partitions of r.*

0.3 Representations of $GL(\mathbb{F}_{p^n})$

Theorem 0.3.1. *The simple $\mathbb{F}_{p^n}GL(\mathbb{F}_{p^n})$-modules are parametrized by 'weights' and are absolutely simple. Thus, for any field k of characteristic p, the simple $kGL(\mathbb{F}_{p^n})$-modules can all be written in $\mathbb{F}_p^n$.*

0.4 Functorial methods

Writing Vec_k for the category of finite dimensional vector spaces over k we consider the category Fun_k of functors $\mathrm{Vec}_k \to \mathrm{Vec}_k$. This is an abelian category. For example, the functor $V \mapsto S^{17}V$ and the functor $V \mapsto \Lambda^6 V$ lie in Fun_k.

Theorem 0.4.1. *The simple objects in Fun_k are parametrized by pairs (n, W) where $n \geq 0$ and W is a simple representation of $GL_n(k)$. The corresponding simple functor sends k^n to W and is zero on spaces of dimension $< n$. Each value on k^m is a simple module for $GL_m(k)$ or zero.*

Chapter 1

Dual spaces and bilinear forms

This follows the first section of James' book and is quite incomplete.

Let M be a finite dimensional vector space over a field k and put $M^* = \mathrm{Hom}_k(M, k)$. Let V be a subspace of M. We may take a basis $e_1, \ldots, e_k$ for V and extend it to a basis $e_1, \ldots, e_m$ for M, so that M^* has a dual basis $\epsilon_1, \ldots, \epsilon_m$ with $\epsilon_i(e_j) = \delta_{ij}$. Thus $\dim M = \dim M^*$.

We put $V^\circ = \{f \in M^* \mid f|_V = 0\}$.

Lemma 1.0.1. *V° has basis $\epsilon_{k+1}, \ldots, \epsilon_m$, so that $\dim V + \dim V^\circ = m$.*

Proposition 1.0.2. *The vector space of bilinear forms $\langle -, - \rangle : M \times M \to k$ is isomorphic to $\mathrm{Hom}(M, M^*)$.*

Proof. Suppose that $\langle -, - \rangle : M \times M \to k$ is a bilinear form on M. There is a map

□

Lemma 1.0.3. *A bilinear form is non-singular if and only if the corresponding map θ is injective. Thus if $\dim M$ is finite, it is equivalent to require that $\theta : M \to M^*$ be an isomorphism.*

We define $V^\perp = \{x \in M \mid \langle x, v \rangle = 0 \text{ for all } v \in V\}$.

Lemma 1.0.4. *If $U \subseteq V$, W are subspaces of M then $V^\perp \subseteq U^\perp$ and $(U + W)^\perp = U^\perp \cap V^\perp$.*

Proposition 1.0.5. *If $\langle -, - \rangle$ is non-singular and $\dim M$ is finite then $\theta : M \to M^*$ maps $V^\perp$ isomorphically to V° and hence $\dim V + \dim V^\perp = \dim M$. We have $V = V^{\perp\perp}$ and $(U \cap V)^\perp = U^\perp + V^\perp$.*

Chapter 2

Representations of S_r

The approach and results are taken directly from James' book.

2.1 Tableaux and tabloids

A *partition* of r is a list of positive integers $\lambda = [\lambda_1, \lambda_2, \lambda_3, \ldots]$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ and $\sum \lambda_i = r$. Partitions are partially ordered by the *dominance* relation: we say that $\lambda = [\lambda_1, \lambda_2, \lambda_3, \ldots]$ dominates $\mu = [\mu_1, \mu_2, \mu_3, \ldots]$ if for all j, $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$.

For each partition we have a *Young subgroup* $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$. The set of right cosets $S_\lambda \backslash S_r$ is a transitive S_r-set whose elements can be described in a certain way, by means of tabloids. We first define a λ-*tableau* to be a Young diagram of shape λ filled with the numbers $\{1, \ldots, r\}$, such as

$$t = \begin{array}{|c|c|c|}\hline 5 & 1 & 2 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

for the partition $\lambda = [3, 2]$. The symmetric group S_r permutes the λ-tableaux by acting on their entries. Let R_t and C_t denote the row and column stabilizers of t, so that in this example

$$R_t = S_{\{5,1,2\}} \times S_{\{4,3\}}, \quad C_t = S_{\{5,4\}} \times S_{\{1,3\}} \times S_{\{2\}}.$$

Lemma 2.1.1. *If t is a λ-tableau and $\pi \in S_r$ then $R_{t\pi} = \pi^{-1} R_t \pi$ and $C_{t\pi} = \pi^{-1} C_t \pi$.*

We put an equivalence relation on λ-tableaux by saying that two are equivalent if the numbers in each row are the same. Thus the equivalence class of t is tR_t. We write the equivalence class as $\{t\}$ and denote it pictorially by leaving out the vertical lines:

$$\{t\} = \begin{array}{ccc}\hline 5 & 1 & 2 \\ \hline 4 & 3 & \\ \cline{1-2}\end{array}$$

Such an equivalence class is called a λ-*tabloid*. The one above is the same as the tabloid

$$\begin{array}{ccc}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array}$$

and evidently we can write tabloids with the entries increasing along each row. Now S_r permutes the set of tabloids by acting on the entries. The stabilizer of the λ-tabloid with the numbers $\{1, \ldots, r\}$ written along the rows in order is the Young subgroup S_λ and the action on the λ-tabloids is transitive. Hence:

Lemma 2.1.2. *The set of λ-tabloids is isomorphic to $S_\lambda \backslash S_r$ as a S_r-set. The number of λ-tabloids is*

$$\frac{r!}{\lambda_1!\lambda_2!\cdots}$$

The S_r-sets that arise this way include examples such as the set of unordered tuples of elements of $\{1, \ldots, r\}$ of some given length, or the set of ordered tuples of some given length, or combinations of these possibilities.

2.2 Permutation modules

Over a ring k define M^λ to be the permutation module on the set of λ-tabloids. Thus $M^\lambda \cong M \uparrow_{S_\lambda}^{S_r}$, and M^λ is generated as a kS_r-module by any single tabloid. If t is a λ-tableau let κ_t be the element of the group algebra kS_r that is the signed sum of the elements of C_t. Thus

$$\kappa_t = \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi.$$

We define $e_t = \{t\}\kappa_t$ as an element of M^λ. In the case of the example

$$t = \begin{array}{|c|c|c|}\hline 5 & 1 & 2 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

we have

$$e_t = \begin{array}{ccc}\hline 5 & 1 & 2 \\ \hline 4 & 3 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \\ \cline{1-2}\end{array}.$$

The *Specht* module S^λ for the partition λ is defined to be the submodule of M^λ spanned by the polytabloids.

Proposition 2.2.1. *Let t be a λ-tableau and $\pi \in S_r$.*

1. *e_t is a linear combination of tabloids with ± 1 coefficients.*

2. *$\kappa_{t\pi} = \pi^{-1}\kappa_t\pi$ and $e_t\pi = e_{t\pi}$.*

3. *S^λ is a kS_r-module and is generated as a kS_r-module by any single polytabloid.*

4. *$\kappa_t = \kappa_{C_1}\kappa_{C_2}\cdots\kappa_{C_s}$ where the C_i are the columns of t.*

Lemma 2.2.2 ('Basic Combinatorial Lemma')**.** *Let λ and μ be partitions of r and suppose that t_1 is a λ-tableau and t_2 is a μ-tableau. Suppose that for every i the numbers from the ith row of t_2 belong to different columns of t_1. Then $\lambda \trianglerighteq \mu$.*

Proof. We may sort the columns of t_1 so that the entries in each column appear in increasing rows of t_2. Now all entries from the first i rows of t_2 appear in the first i rows of t_1, for each i. Then the number of entries in the first i rows of t_2 namely, $\mu_1 + \mu_2 + \cdots + \mu_i$ is less that the number of entries in the first i rows of t_1. $\square$

Lemma 2.2.3. *Let λ and μ be partitions of r. Suppose that t is a λ-tableau and t^* is a μ-tableau, and that $\{t^*\}\kappa_t \neq 0$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then $\{t^*\}\kappa_t = \pm\{t\}\kappa_t = \pm e_t$.*

Proof. There are two possibilities:
(a) for every row of t^* the entries lie in different columns of t, or
(b) there exist numbers a and b in the same row of t^* and same column of t. (a) implies $\lambda \trianglerighteq \mu$ by Lemma 2.2.2. Assuming (b) we have $\{t^*\} = \{t^*\}(a, b)$. Also $(1 - (a, b))$ is a factor of κ_{C_i} if a, b lie in column i of t, and this is a factor of κ_t. This shows that $\{t^*\}\kappa_t = 0$ in this case.

Now suppose $\lambda = \mu$. The possibility $\{t^*\}\kappa_t = 0$ is excluded by hypothesis, so we are in case (a) and we may write $\{t^*\} = \{t\}\pi$ for some $\pi \in C_t$ by rearranging the columns. Thus $\{t^*\}\kappa_t = \{t\}\pi\kappa_t = \pm\{t\}\kappa_t = \pm e_t$ $\square$

Corollary 2.2.4. *If u is an element of M^μ and t is a μ-tableau, then $u\kappa_t$ is a multiple of e_t.*

Proof. This is because u is a linear combination of μ-tabloids $\{t^*\}$ and $\{t^*\}\kappa_t$ is a multiple of e_t always, by Lemma 2.2.3. $\square$

Let $\langle -, - \rangle$ be the standard bilinear form on M^λ with respect to its permutation basis. This form is S_r-invariant.

Lemma 2.2.5. *We have $\langle u\kappa_t, v \rangle = \langle u, v\kappa_t \rangle$.*

Proof. Calculate. $\square$

Theorem 2.2.6 (The Submodule Theorem, James)**.** *If U is a kS_r-submodule of M^μ then either $U \supseteq S^\mu$ or $U \subseteq S^{\mu\perp}$.*

Proof. Suppose $u \in U$ and t is a μ-tableau. Then by Corollary 2.2.4 $u\kappa_t$ is a multiple of e_t. There are two possibilities
(a) $u\kappa_t = 0$ always, (b) we can find u and t so that $u\kappa_t \neq 0$.

In case (a) we have $0 = \langle u\kappa_t, \{t\} \rangle = \langle u, \{t\}\kappa_t \rangle = \langle u, e_t \rangle$ always, so that $U \subseteq S^{\mu\perp}$ since the e_t span S^μ. In case (b) we have that $u\kappa_t$ is a non-zero multiple of e_t, and this belongs to U. Since S^μ is generated by any single e_t we deduce that $U \supseteq S^\mu$. $\square$

We say that a kG-module V is *absolutely irreducible* if $V \otimes_k K$ is an irreducible KG-module for all extension fields $K \supseteq k$.

Theorem 2.2.7. *$S^\mu/(S^\mu \cap S^{\mu\perp})$ is either zero or aboslutely irreducible. Furthermore, if this is non-zer, then $S^\mu \cap S^{\mu\perp}$ is the unique maximal submodule of S^μ, and $S^\mu/(S^\mu \cap S^{\mu\perp})$ is self-dual.*

Proof. James page 15. $\square$

In characteristic zero $S^\mu \cap S^{\mu\perp} = 0$ always, so the S^μ are irreducible. We will conclude that we have a complete list of irreducible kS_r-modules after showing that the S^μ are all non-isomorphic, since the number of irreducible representations equals the number of conjugacy classes of S_r, which equals the number of partitions of r, which equalis the number of isomorphism classes of Specht modules.

Example 2.2.8. Take $\lambda = [n-1, 1]$. If the characteristic of k is p then $p \not| \, r$ if and only if $M^\lambda = S^\lambda \oplus S^{\lambda\perp} = S^{[r-1,1]} \oplus S^{[r]}$, if and only if $S^{[r-1,1]}$ is irreducible. On the other hand $p \mid n$ if and only if $0 \subset S^{\lambda\perp} \subset S^\lambda \subset M^\lambda$ is the unique composition series of M^λ, and then $S^\lambda / S^{\lambda\perp}$ is irreducible.

This is because S^λ is the coordinate sum zero subspace of M^λ, so that $(S^\lambda)^\perp$ is spanned by the vector with 1 in every coordinate, which lies in S^λ if and only if $p \mid r$. In that case, because every submodule of M^λ either contains S^λ or is contained in $(S^\lambda)^\perp$, these are the only proper submodules of M^λ.

Lemma 2.2.9. *Let $\theta : M^\lambda \to M^\mu$ be a kS_r-homomorphism and suppose that $S^\lambda \not\subseteq \operatorname{Ker}\theta$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ the restriction of θ to S^λ is multiplication by a constant.*

Proof. James page 16. $\square$

Corollary 2.2.10. *If k has characteristic zero and $\theta : S^\lambda \to M^\mu$ is non-zero then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then θ is multiplication by a constant.*

Proof. This follows because in characteristic zero $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$, so that any homomorphism θ extends to a homomorphism from M^λ by defining it to be zero on $(S^\lambda)^\perp$. $\square$

Theorem 2.2.11. *The Specht modules over $\mathbb{Q}$ are self-dual and absolutely irreducible, and give all the ordinary irreducible representations of S_r.*

2.2.1 Exercise

1. Suppose V has a unique maximal submodule V_1. Show that V/V_1 is simple. Let $\theta : V \to M$ be a nonzero homomorphism. Show that M has a composition factor isomorphic to V/V_1.

2.3 p-regular partitions

We say that a partition $\lambda \vdash r$ is *p-regular* if λ has $< p$ parts of each size, and otherwise λ is *p-singular.* For example if $p = 2$ and $r = 5$ the 2-regular partitions are $[5], [4, 1], [3, 2]$. Our goal is to show that $S^\lambda/(S^\lambda \cap S^{\lambda\perp} \neq 0$ if and only if λ is p-regular, and that this gives a complete list of irreducible kS_r-modules over any field of characteristic p.

We define an element g of a finite group G to be p-regular if and only if its order is not divisible by p. We will use the fact that, over a large enough field k of characteristic p, the number of irreducible kG-modules equals the number of p-regular classes of G.

Lemma 2.3.1. *An element $\pi \in S_r$ is p-regular if and only if π has no cycles of length divisible by p, if and only if the cycle type of π has no parts divisible by p.*

Proof. This is clear. $\square$

Lemma 2.3.2. *The number of p-regular partitions of r equals the number of p-regular conjugacy classes of S_r.*

Proof. We simplify the expression

$$\frac{(1-x^p)(1-x^{2p})\cdots}{(1-x)(1-x^2)\cdots}$$

in two ways. First we cancel all the terms in the numerator with the corresponding terms in the denominator, to give a product

$$\prod_{p\nmid i} \frac{1}{(1-x^i)} = \prod_{p\nmid i}(1 + x^i + (x^i)^2 + \cdots).$$

In this product the coefficient of x^r is the number of partitions of r where no parts are divisible by p, the partition $[\ldots 3^c 2^b 1^a]$ corresponding to a term with x^a from the first bracket, $(x^2)^b$ from the second, $(x^3)^c$ from the third, and so on.

In the second way of canceling terms, we factor each term of the denominator into the term of the numerator immediately above it, giving

$$\prod_{m=1}^{\infty}(1 + x^m + \cdots + (x^m)^{p-1}).$$

Now the coefficient of x^r is the number of partitions with no part occurring p or more times, using a similar correspondence with partitions to the one described before. $\square$

We define $g^\mu = \gcd\{\langle e_t, e_{t^*}\rangle \mid t, t^* \text{ range over all } \mu\text{-polytabloids}\}$.

Lemma 2.3.3. *Over $\mathbb{F}_p$, $S^\mu/(S^\mu \cap S^{\mu\perp}) = 0$ if and only if $S^\mu \subseteq S^{\mu\perp}$, if and only if $p \mid g^\mu$.*

In the following we will let $\mu = [\ldots, 3^{a_3}, 2^2, 1^{a_1}]$, so that a_j is the number of parts of μ of size j.

Proposition 2.3.4. $\prod_{j=1}^{\infty} a_j! \mid g^\mu$ *and* $g^\mu \mid \prod_{j=1}^{\infty}(a_j!)^j$.

Corollary 2.3.5. *Over $\mathbb{F}_p$, $S^\mu/(S^\mu \cap S^{\mu\perp}) \neq 0$ if and only if μ is p-regular.*

When μ is p-regular we define $D^\mu = S^\mu/(S^\mu \cap S^{\mu\perp})$, and this is an irreducible, self-dual kS_r-module. We wish to show that this accounts for all irreducible kS_r-modules, and for this we need to know that if $\lambda \neq \mu$ then $D^\mu \ncong D^\lambda$.

Lemma 2.3.6. *Let $\theta : S^\lambda \to M^\mu/U$ be a non-zero homomorphism of kS_r-modules, where U is some submodule of M^μ. Then $\lambda \trianglerighteq \mu$; and $\lambda = \mu$ implies that $\operatorname{Im}\theta \subseteq (S^\mu + U)/U$.*

Corollary 2.3.7. *Suppose that $\theta : D^\lambda \to M^\mu/U$ is a non-zero homomorphism, where λ is p-regular and $U \subseteq M^\mu$. Then $\lambda \trianglerighteq \mu$, and $U \supseteq S^\mu$ implies that $\lambda \triangleright \mu$.*

Theorem 2.3.8 (James)**.** *Let k have characteristic p. The D^μ, where μ is p-regular, form a complete set of irreducible kS_r-modules. Furthermore $D^\mu \cong D^{\mu *}$ and D^μ is absolutely irreducible. Every field is a splitting field for S_r.*

Theorem 2.3.9 (James)**.** *All composition factors of M^μ have the form D^λ with $\lambda \triangleright \mu$, except if μ is p-regular, in which case D^μ also occurs as a composition factor with multiplicity 1.*

2.4 Young modules

We will use the Krull-Schmidt theorem in what follows.

Proposition 2.4.1. *Let λ be a partition of r and write $M^\lambda = Y_1 \oplus \cdots Y_d$ as a direct sum of indecomposable $\mathbb{F}_p S_r$-modules. There is a unique i with $Y_i \supseteq S^\lambda$. We write Y^λ for this summand. Then Y^λ is determined independently of the direct sum decomposition. Furthermore, $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$.*

Proof. We apply the submodule theorem of James: for each i, either $U_i \supseteq S^\lambda$ or $U_i \subseteq S^{\lambda\perp}$. Since not all U_i can be contained in $S^{\lambda\perp}$ there must be at least one that is not, and there can be no more than one since two such summands would not intersect in 0.

To show that Y^λ is defined independently of the decomposition, we use the fact that permutation modules and all homomorphisms between them are liftable from $\mathbb{F}_p$ to $\mathbb{Z}_p$. This comes about because there is a basis for homomorphisms between permutation modules given in terms of double cosets, valid over any ring, so that homomorphisms lift. Furthermore, idempotents lift to idempotents. This means that a decomposition $M^\lambda = Y_1 \oplus \cdots Y_d$ is the reduction of a decomposition $\tilde{M}^\lambda = \tilde{Y}_1 \oplus \cdots \tilde{Y}_d$ of $\mathbb{Z}_p S_r$-modules. By the same argument as above applied to $\mathbb{Q}_p \otimes \tilde{M}^\lambda$, one of the summands $\mathbb{Q}_p \otimes \tilde{Y}_j$ contains the Specht module $\mathbb{Q}_p \otimes \tilde{S}^\lambda$, and j must be the previous i because the reduction of $\tilde{Y}_j$ modulo p, namely Y_j must contain the reduction of $\tilde{S}^\lambda$ modulo p, namely S^λ. Since $\mathbb{Q}_p \otimes \tilde{S}^\lambda$ occurs with multiplicity 1 in $\mathbb{Q}_p \otimes \tilde{M}^\lambda$ it follows that Y_i is the reduction of the unique summand $\tilde{Y}_i$ for which $\mathbb{Q}_p \otimes \tilde{Y}_i$ has $\mathbb{Q}_p \otimes \tilde{S}^\lambda$ as a composition factor, and this determines the isomorphism type of $\tilde{Y}_i$. Since Y_i is the reduction modulo p, its isomorphism type is also determined and it shows that if $\mu \neq \lambda$ then $Y^\mu \ncong Y^\lambda$.

The other composition factors of $\mathbb{Q}_p \otimes \tilde{M}^\lambda$ are all Specht modules $\mathbb{Q}_p \otimes \tilde{S}^\mu$ with $\mu \trianglerighteq \lambda$, and so if any of the other summands of M^λ are Young modules, they must be Y^μ with $\mu \trianglerighteq \lambda$ □

In fact, all of the summands of the M^λ are Young modules, but we will deduce this from information coming from the Schur algebra. For now, we will take this as a hypothesis that needs to be proved.

Theorem 2.4.2. *1. The Y^λ are self-dual.*

2. Y^λ appears as a summand of M^λ with multiplicity 1.

3. If Y^μ is a summand of M^λ then $\mu \trianglerighteq \lambda$.

Proof. Evidently $(Y^\lambda)^*$ is a summand of $(M^\lambda)^* \cong M^\lambda$ and its lift $\mathbb{Q}_p \otimes \tilde{Y}^{\lambda *}$ has $\mathbb{Q}_p \otimes \tilde{S}^{\lambda *} \cong \mathbb{Q}_p \otimes \tilde{S}^\lambda$ as a composition factor. This identifies $(Y^\lambda)^*$ as Y^λ. Since Y^λ is characterized as the unique summand of M^λ for which $\mathbb{Q}_p \otimes \tilde{Y}^\lambda$ has $\mathbb{Q}_p \otimes \tilde{S}^\lambda$ as a composition factor, it occurs with multiplicity 1 in M^λ. Since the other composition factors of $\mathbb{Q}_p \otimes \tilde{M}^\lambda$ are $\mathbb{Q}_p \otimes \tilde{S}^\mu$ with $\mu \triangleright \lambda$, any other summands Y^μ of M^λ have $\mu \triangleright \lambda$. □

Chapter 3

The Schur algebra

3.1 Tensor space

Let E be a vector space of dimension n over k. Elements of $E^{\otimes r}$ are linear combinations of tensors $v_1 \otimes \cdots \otimes v_r$ where $v_i \in E$ and the symmetric group S_r acts on these by permuting the positions in which vectors from E occur in the tensor product. Thus when $r = 3$, for example,

$$(v_1 \otimes v_2 \otimes v_3)(1,2) = v_2 \otimes v_1 \otimes v_3$$

and

$$(v_1 \otimes v_2 \otimes v_3)(1,2)(2,3) = v_2 \otimes v_3 \otimes v_1 = (v_1 \otimes v_2 \otimes v_3)(1,3,2).$$

We see that

$$(v_1 \otimes v_2 \otimes v_3 \cdots)\pi = v_{1\pi^{-1}} \otimes v_{2\pi^{-1}} \otimes v_{3\pi^{-1}} \cdots$$

gives a right action of S_r on $E^{\otimes r}$.

Proposition 3.1.1. *As a kS_r-module, $E^{\otimes r}$ is a permutation module, and is a direct sum of modules M^λ where λ is a partition of r with at most n parts. Every such M^λ appears as a summand of $E^{\otimes r}$. Thus $E^{\otimes r}$ is a direct sum of Young modules Y^λ where λ is a partition of r with at most n parts.*

Proof. Let $e_1, \ldots, e_r$ be a basis for E. The basic tensors $e_{i_1} \otimes \cdots \otimes e_{i_r}$ are permuted by S_r and each orbit contains a tensor $\cdots e_{i_2} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1} \otimes \cdots \otimes e_{i_1}$ where the λ_j suffices all equal to i_j are grouped together, and we may assume $\cdots \geq \lambda_2 \geq \lambda_1$. Thus $\lambda = [\cdots \lambda_2, \lambda_1]$ is a partition of r, and it has at most n parts, because there are at most n possible values for the λ_i. The stabilizer of such a tensor is the Young subgroup S_λ. Thus the tensors in this orbit biject with the cosets $S_\lambda \backslash S_r$ and span a copy of M^λ, where λ has at most n parts. The final statement about Young modules follows from the decomposition of the M^λ into Young modules. $\square$

We define the Schur algebra $S_k(n,r)$ to be $\mathrm{End}_{kS_r}(E^{\otimes r})$. Looking ahead to our application of this, we let the general linear group $GL(E)$ act diagonally on $E^{\otimes r}$, so that if $g \in GL(E)$ then $g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r$. Evidently this commutes with the action of S_r, so we get a ring homomorphism $kGL(E) \to S(n,r)$, and $S(n,r)$-modules become $kGL(E)$-modules.

Example 3.1.2. Let $r = 2$ and $\dim E = 2$. Then E has a basis e_1, e_2, and $E^{\otimes 2}$ has a basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. This is permuted by S_2 with orbits $\{e_1 \otimes e_1\}, \{e_1 \otimes e_2, e_2 \otimes e_1\}, \{e_2 \otimes e_2\}$ so that $E^{\otimes 2} \cong M^{[2]} \oplus M^{[1^2]} \oplus M^{[2]}$.

In characteristic 2 these three modules are indecomposable and are equal to the corresponding Young modules. This is apparent for $M^{[2]}$ because it has dimension 1. We can see that $M^{[1^2]}$ is indecomposable in various ways. One way, consistent with the theory of representations of the symmetric groups, is to observe that $S^{[1^2]}$ is spanned by the polytabloid

$$\begin{array}{c} \hline 1 \\ \hline 2 \\ \hline \end{array} - \begin{array}{c} \hline 2 \\ \hline 1 \\ \hline \end{array}$$

so it has dimension 1 and is self-dual in characteristic 2, so $S^{[1^2]\perp} = S^{[1^2]}$. By the submodule theorem, all submodules of $M^{[1^2]}$ either contain or are contained in this space, so it is the only proper submodule of $M^{[1^2]}$. This forces $M^{[1^2]}$ to be indecomposable and we may describe it pictorially as $\substack{k\\k}$. Thus $S(2,2) = \mathrm{End}(k \oplus \substack{k\\k} \oplus k)$. Let us describe the algebra $S(2,2)^\diamond := \mathrm{End}(k \oplus \substack{k\\k})$ which is related to $S(2,2)$ as $S(2,2)^\diamond = eS(2,2)e$ where e is projection onto $k \oplus \substack{k\\k}$. In order to understand this relationship fully we should study Morita equivalence.

We may write elements of $S(2,2)^\diamond$ as matrices

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

where $\phi_{11} \in \mathrm{Hom}(k,k)$, $\phi_{12} \in \mathrm{Hom}(\substack{k\\k}, k)$, $\phi_{21} \in \mathrm{Hom}(k, \substack{k\\k})$, $\phi_{22} \in \mathrm{Hom}(\substack{k\\k}, \substack{k\\k})$, and such matrices multiply together with the usual rules of matrix multiplication. These four spaces have dimensions $1,1,1,2$, the first spanned by 1_k, the last spanned by $1_{\substack{k\\k}}$ and another nilpotent endomorphism. Thus $S(2,2)^\diamond$ can be realized as the set of matrices

$$\begin{bmatrix} a & 0 & f \\ g & b & h \\ 0 & 0 & b \end{bmatrix}.$$

It can be easier to understand this algebra pictorially. There are two simple modules α, β where 1_k acts as 1 on α and 0 on β, and $1_{\substack{k\\k}}$ acts as 1 on β and as 0 on α. These two idempotents give a decomposition of $S(2,2)^\diamond$ as a direct sum of

$$S(2,2)^\diamond 1_k = S(2,2)^\diamond \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{ \begin{bmatrix} a & 0 & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \}$$

and

$$S(2,2)^\diamond 1_{\substack{k\\k}} = S(2,2)^\diamond \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \{ \begin{bmatrix} 0 & 0 & f \\ 0 & b & h \\ 0 & 0 & b \end{bmatrix} \}$$

as left $S(2,2)^\diamond$-modules. These modules are projective because they are summands of the regular representation. We see that their structure is

$$S(2,2)^\diamond = \substack{\alpha\\\beta} \oplus \substack{\beta\\\alpha\\\beta}$$

where we identify the composition factors by the action of the idempotents 1_k and $1_{\substack{k\\k}}$. We may now list the five indecomposable modules of this algebra. We see also that it has finite global dimension and write down projective resolutions of the simples α and β.

If we had done a similar calculation with $S(2,2)$ instead of $S(2,2)^\diamond$ the difference would have been that in writing down matrices for the endomorphisms of $k \oplus k \oplus \substack{k\\k}$ there would have been have been a 2×2 block in the top left corner, giving a matrix summand with a single simple module, now 2-dimensional instead of 1-dimensional. The submodule structure of the projective modules would have been described by the same pictures.

We may also see explicitly what $E^{\otimes 2}$ looks like as an $S_{\mathbb{F}_2}(2,2)$-module.

Proposition 3.1.3. *As an $S_{\mathbb{F}_2}(2,2)$-module, $E^{\otimes 2} = \substack{\beta\\\alpha\\\beta}$ is uniserial, with $\substack{\alpha\\\beta} = \mathrm{ST}^2(E)$ (the symmetric tensors), $\substack{\beta\\\alpha} = \mathrm{Sym}^2(E)$ (the symmetric power) and $\beta = \Lambda^2(E)$.*

Proof. Regarding $E = k^2 \oplus \substack{k\\k} = y \oplus \substack{x\\z}$ we have a composition series $z \subseteq y + z \subseteq x+y+z$ for the action of $S_{\mathbb{F}_2}(2,2)$, and the action of the idempotents 1_{k^2} and $1_{\substack{k\\k}}$ shows that the composition factors are β, α, β taken in order. Furthermore, consideration of the endomorphisms shows that this is the only composition series. From earlier identification of these terms, we see that this series has terms with bases $e_1 \otimes e_2 + e_2 \otimes e_1$ and $e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_1, e_2 \otimes e_2$ this identifies the terms as claimed. $\square$

We may also example the action of elements of $GL(E)$ on these subspaces by writing the elements as matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with respect to the basis e_1, e_2, and we find that the action of such matrix is via another matrix with entries that are homogeneous polynomials of degree 2 in a, b, c, d. In the case of the module α the matrix is $\begin{bmatrix} a^2 & b^2 \\ c^2 & d^2 \end{bmatrix}$ which describes a Frobenius twist of the natural representation of $GL(E)$.

When the characteristic of k is not 2 we have decompositions $M^{[1^2]} = S^{[2]} \oplus S^{[1^2]}$ and $E^{\otimes 2} = (S^{[2]})^3 \oplus S^{[1^2]}$. This module is semisimple and so $S(2,2) = M_3(k) \oplus k$ as a ring. It has two simple modules, of dimensions 3 and 1. Now $E^{\otimes 2} = \mathrm{Sym}^2(E) \oplus \Lambda^2(E)$ and these subspaces have bases $e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_1, e_2 \otimes e_2$ and $e_1 \otimes e_2 - e_2 \otimes e_1$.

3.2 Homomorphisms between permutation modules

The next results are what was needed in identifying the properties of Young modules.

Lemma 3.2.1. *Let G permute a set Ω and let H be a subgroup of G and let $\omega \in \Omega$. Then there is a map of G-sets $H\backslash G \to \Omega$ which sends $Hg \mapsto \omega g$ if and only if $H \subseteq \mathrm{Stab}_G(\omega)$.*

Proof. The specification given is a morphism of G-sets provided that it is well-defined, which means that if $Hg_1 = Hg$ then $\omega g_1 = \omega g$. In such a case $g_1 = hg$ for some $h \in H$, so that $\omega g_1 = \omega hg$ and this equals ωg if and only if $\omega h = \omega$ or, in other words, $h \in \mathrm{Stab}_G(\omega)$. □

Proposition 3.2.2. *Let G be a finite group with subgroups H and K and let R be a commutative ring. Then $\mathrm{Hom}_{RG}(R[H\backslash G], R[K\backslash G])$ is a free R-module with basis in bijection with $K\backslash G/H$. Specifically, to each double coset KgH there corresponds a RG-homomorphism $H \mapsto \sum_{h\in[K\backslash KgH]} Kgh = \sum_{h\in[(K^g\cap H)\backslash H]} Kgh$.*

Proof. The expression $\sum_{h\in[K\backslash KgH]}$ is the sum of the elements in the H-orbit containing $K \in R[K\backslash G]$, and it is stabilized by H. Therefore there is a map of G sets from $H\backslash G$ to $R[K\backslash G]$ sending H to $\sum_{h\in[K\backslash KgH]}$ and extends to an RG-module homomorphism defined on $R[H\backslash G]$. The morphisms constructed in this way are all linearly independent, because their images have disjoint support. The two sums are equal because $K\backslash KgH \cong (K^g \cap H)\backslash H$ as H-sets. It remains to show that these morphisms span the full space of homomorphisms. One can argue that the image of H under any homomorphism must have coefficients that are constant on H-orbits. One can also use Frobenius reciprocity to show that $\mathrm{Hom}_{RG}(R[H\backslash G], R[K\backslash G]) \cong \bigoplus_{K\backslash G/H} \mathrm{Hom}(R, R)$, which has rank equal to $|K\backslash G/H|$. □

Corollary 3.2.3. *Permutation modules and homomorphisms between them lift through ring homomorphisms.*

Proof. This is because each G-set is uniquely a disjoint union of orbit, giving a uniquely determined direct sum decomposition of the corresponding permutation module. The formula for the basis of homomorphisms between two summands is independent of the choice of ground ring. □

Proposition 3.2.4. *If $\lambda, \mu \vdash r$ then double cosets $S_\lambda\backslash S_r/S_\mu$ biject with row-semistandard λ-tableaux of type μ.*

Proof. Needed. □

The next result uses notation θ_T of James, not so far introduced. Let $e \in S_F(n, r)$ be the endomorphism of $E^{\otimes r}$ that is projection onto a summand $M^{[1^r]}$, spanned by the $e_{u,u\pi}$, $\pi \in S_r$, $u = (1, 2, \ldots, r)$.

Exercises need to be written. Should I give the homomorphism in 3.2.2 a name such as θ_g for future reference?

Proposition 3.2.5. *$S_F(n,r)e$ has basis the θ_T where T is a $[1^r]$-tableau of content μ, and μ is a composition of r with $\leq n$ parts. The mapping $S(n,r)e \to E^{\otimes r}$ given by $\theta_T \mapsto e_T$ is an isomorphism of $S_F(n,r)$-modules.*

Proof. e is the identity on $M^{[1^r]}$ and 0 on the others summands. Therefore $\theta_T e \in S_F(n,r)e$ is zero unless T is a $[1^r]$-tableau of content μ. For such T, $\theta_T e = \theta_T$, and so these θ_T are independent. □

3.3 The structure of endomorphism rings

Be specific about right and left modules.

Proposition 3.3.1. *Let U be a module for a ring A with a 1. Expressions*

$$U = U_1 \oplus \cdots \oplus U_n$$

as a direct sum of submodules biject with expressions $1_U = e_1 + \cdots + e_n$ for the identity $1_U \in \mathrm{End}_A(U)$ as a sum of orthogonal idempotents. Here e_i is obtained from U_i as the composite of projection and inclusion $U \to U_i \to U$, and U_i is obtained from e_i as $U_i = e_i(U)$. The summand U_i is indecomposable if and only if e_i is primitive.

Proof. We must check several things. Two constructions are indicated in the statement of the proposition: given a direct sum decomposition of U we obtain an idempotent decomposition of 1_U, and vice-versa. It is clear that the idempotents constructed from a module decomposition are orthogonal and sum to 1_U. Conversely, given an expression $1_U = e_1 + \cdots + e_n$ as a sum of orthogonal idempotents, every element $u \in U$ can be written $u = e_1 u + \cdots + e_n u$ where $e_i u \in e_i U = U_i$. In any expression $u = u_1 + \cdots u_n$ with $u_i \in e_i U$ we have $e_j u_i \in e_j e_i U = 0$ if $i \neq j$ so $e_i u = e_i u_i = u_i$, and this expression is uniquely determined. Thus the expression $1_U = e_1 + \cdots + e_n$ gives rise to a direct sum decomposition.

We see that U_i decomposes as $U_i = V \oplus W$ if and only if $e_i = e_V + e_W$ can be written as a sum of orthogonal idempotents, and so U_i is indecomposable if and only if e_i is primitive. □

Corollary 3.3.2. *An A-module U is indecomposable if and only if the only non-zero idempotent in $\mathrm{End}_A(U)$ is 1_U.*

Proof. From the proposition, U is indecomposable if and only if 1_U is primitive, and this happens if and only if 1_U and 0 are the only idempotents in $\mathrm{End}_A(U)$. This last implication in the forward direction follows since any idempotent e gives rise to an expression $1_U = e + (1_U - e)$ as a sum of orthogonal idempotents, and in the opposite direction there simply are no non-trivial idempotents to allow us to write $1_U = e_1 + e_2$. □

The next theorem might be called the 'Fundamental Theorem of Endomorphism Rings'. If U is a right A-module and we let endomorphisms of U act from the left, U becomes a $(\mathrm{End}_A(U), A)$-bimodule. We have functors between right A-modules and left $\mathrm{End}_A(U)$-modules as follows: if M is a right A-module we put $M^\natural := \mathrm{Hom}_A(M, U)$, and if W is a left $\mathrm{End}_A(U)$-module we put $W^\natural := \mathrm{Hom}_{\mathrm{End}_A(U)}(W, U)$.

Theorem 3.3.3. *Let U be a module for a ring A with a 1. There is a contravariant equivalence of categories between the full subcategory of A-modules with objects the direct sums of summands of U, and the full subcategory of $\mathrm{End}_A(U)$-modules with objects the direct sums of modules $\mathrm{End}_A(U)e$, where $e^2 = e \in \mathrm{End}_A(U)$.*

The $\mathrm{End}_A(U)$-modules $\mathrm{End}_A(U)e$ are all projective, but in general not all projective $\mathrm{End}_A(U)$-modules are direct sums of these. When A is a finite dimensional algebra over a field the projective modules are all of this form, but we have not proven this yet.

Proof. Write the first category as $\mathrm{Add}(U)$ and the second category as $\mathrm{Proj}_e(\mathrm{End}(U))$. We define a functor

$$\mathrm{Add}(U) \to \mathrm{Proj}_e(\mathrm{End}(U))$$

by $M \mapsto \mathrm{Hom}_A(M, U)$, and a functor

$$\mathrm{Proj}_e(\mathrm{End}(U)) \to \mathrm{Add}(U)$$

by $P \mapsto \mathrm{Hom}_{\mathrm{End}(U)}(P, U)$. We verify that if $U_i = e_i(U)$ is a summand of U, then $\mathrm{Hom}_A(U_i, U) \cong \mathrm{End}(U)e_i$ and that if e is an idempotent in $\mathrm{End}(U)$ then

$$\mathrm{Hom}_{\mathrm{End}(U)}(\mathrm{End}(U)e, U) \cong eU$$

as A-modules. □

Corollary 3.3.4. *Let R be a field. For each partition λ of r into at most n parts there is an indecomposable projective $S_R(n, r)$-module P_λ, with distinct partitions giving non-isomorphic modules.*

When R is a field it is a fact that every indecomposable projective $S_R(n, r)$-module has the form $S_R(n, r)e$ for some idempotent element e, but we do not know this yet.

Corollary 3.3.5. *Let λ be a partition of r with at most n parts. The left regular representation of $S_R(n, r)$ is isomorphic to a direct sum of modules $ST^\lambda(E) := ST^{\lambda_1}(E) \otimes \cdots \otimes ST^{\lambda_d}(E)$, and these are projective $S_R(n, r)$-modules. Every indecomposable projective $S_R(n, r)$-module of the form $S_R(n, r)e$ is a direct summand of one of these. In particular, the symmetric tensors $ST^r(E)$ is projective as a $S_R(n, r)$-module, as is $E^{\otimes r}$.*

Proof. We apply the functor to the permutation module M^λ, which is a direct summand of $E^{\otimes r}$ as an $S_R(n, r)$-module. Now

$$\begin{aligned}
\mathrm{Hom}_{RS_r}(M^\lambda, E^{\otimes r}) &= \mathrm{Hom}_{RS_r}(R \uparrow_{S_\lambda}^{S_r}, E^{\otimes r}) \\
&\cong \mathrm{Hom}_{RS_\lambda}(R, E^{\otimes r} \downarrow_{S_\lambda}^{S_r}) \\
&\cong (E^{\otimes \lambda_1})^{S_{\lambda_1}} \otimes \cdots \otimes (E^{\otimes \lambda_d})^{S_{\lambda_d}} \\
&= ST^{\lambda_1}(E) \otimes \cdots \otimes ST^{\lambda_d}(E).
\end{aligned}$$

□

Example 3.3.6. When $R = \mathbb{F}_2$, $n = r = 2$ we have $ST^{[2]} = \begin{smallmatrix}\alpha\\\beta\end{smallmatrix}$ and $ST^{[1^2]} = E^{\otimes 2} = \begin{smallmatrix}\beta\\\alpha\\\beta\end{smallmatrix}$. Both modules have been seen to be projective.

Corollary 3.3.7. *When F is a field of characteristic 0, or p where $p > r$, the functors $\natural$ given contravariant equivalences between FS_r -mod and $S_F(n,r)$ -mod.*

Proof. In this situation FS_r is semisimple and hence so is $S_F(n,r)$. It follows that the equivalence provided by the functors $\natural$ is between the full module categories. □

Expand. Say that $S_F(n,r)$ is semisimple, and identify the Weyl modules, and $E^{\otimes r}$ as a bimodule, namely a direct sum of tensor projects of Weyl and Specht modules.

3.4 The radical

The next lines are taken from my book.

We put

$$\operatorname{Rad} U = \bigcap\{M \mid M \text{ is a maximal submodule of } U\}.$$

In our applications U will always be Noetherian, so provided $U \neq 0$ this intersection will be non-empty and hence $\operatorname{Rad} U \neq U$. If U has no maximal submodules (for example, if $U = 0$, or in more general situations than we consider here where U might not be Noetherian) we set $\operatorname{Rad} U = U$.

Lemma 3.4.1. *Let U be a module for a ring A.*

(1) Suppose that $M_1, \dots, M_n$ are maximal submodules of U. Then there is a subset $I \subseteq \{1, \dots, n\}$ such that

$$U/(M_1 \cap \cdots \cap M_n) \cong \bigoplus_{i \in I} U/M_i$$

which, in particular, is a semisimple module.

(2) Suppose further that U has the descending chain condition on submodules. Then $U/\operatorname{Rad} U$ is a semisimple module, and $\operatorname{Rad} U$ is the unique smallest submodule of U with this property.

Proof. (1) Let I be a subset of $\{1, \dots, n\}$ maximal with the property that the quotient homomorphisms $U/(\bigcap_{i\in I} M_i) \to U/M_i$ induce an isomorphism $U/(\bigcap_{i\in I} M_i) \cong \bigoplus_{i\in I} U/M_i$. We show that $\bigcap_{i\in I} M_i = M_1 \cap \cdots \cap M_n$ and argue by contradiction. If it were not the case, there would exist M_j with $\bigcap_{i\in I} M_i \not\subseteq M_j$. Consider the homomorphism

$$f : U \to (\bigoplus_{i\in I} U/M_i) \oplus U/M_j$$

whose components are the quotient homomorphisms $U \to U/M_k$. This has kernel $M_j \cap \bigcap_{i\in I} M_i$, and it will suffice to show that f is surjective, because this will imply that the larger set $I \cup \{j\}$ has the same property as I, thereby contradicting the maximality of I.

To show that f is surjective let $g : U \to U/\bigcap_{i\in I} M_i \oplus U/M_j$ and observe that $(\bigcap_{i\in I} M_i)+M_j = U$ since the left-hand side is strictly larger than M_j, which is maximal in U. Thus if $x \in U$ we can write $x = y + z$ where $y \in \bigcap_{i\in I} M_i$ and $z \in M_j$. Now $g(y) = (0, x + M_j)$ and $g(z) = (x + \bigcap_{i\in I} M_i, 0)$ so that both summands $U/\bigcap_{i\in I} M_i$ and U/M_j are contained in the image of g and g is surjective. Since f is obtained by composing g with the isomorphism that identifies $U/\bigcap_{i\in I} M_i$ with $\bigoplus_{i\in I} U/M_i$, we deduce that f is surjective.

(2) By the assumption that U has the descending chain condition on submodules, $\operatorname{Rad} U$ must be the intersection of finitely many maximal submodules. Therefore $U/\operatorname{Rad} U$ is semisimple by part (1). If V is a submodule such that U/V is semisimple, say $U/V \cong S_1 \oplus \cdots \oplus S_n$ where the S_i are simple modules, let M_i be the kernel of $U \to U/V \xrightarrow{\text{proj.}} S_i$. Then M_i is maximal and $V = M_1 \cap \cdots \cap M_n$. Thus $V \supseteq \operatorname{Rad} U$, and $\operatorname{Rad} U$ is contained in every submodule V for which U/V is semisimple. □

We define the *radical* of a ring A to be the radical of the regular representation $\operatorname{Rad}{}_A A$ and write simply $\operatorname{Rad} A$. We present some identifications of the radical that are very important theoretically, and also in determining what it is in particular cases.

Proposition 3.4.2. *Let A be a ring. Then,*

(1) $\operatorname{Rad} A = \{a \in A \mid a \cdot S = 0$ *for every simple A-module S*$\}$, *and*

(2) $\operatorname{Rad} A$ *is a 2-sided ideal of A.*

(3) Suppose further that A is a finite dimensional algebra over a field. Then

- *(a)* $\operatorname{Rad} A$ *is the smallest left ideal of A such that $A/\operatorname{Rad} A$ is a semisimple A-module,*
- *(b) A is semisimple if and only if* $\operatorname{Rad} A = 0$,
- *(c)* $\operatorname{Rad} A$ *is nilpotent, and is the largest nilpotent ideal of A.*
- *(d)* $\operatorname{Rad} A$ *is the unique ideal U of A with the property that U is nilpotent and A/U is semisimple.*

Proof. (1) Given a simple module S and $0 \neq s \in S$, the module homomorphism ${}_A A \to S$ given by $a \mapsto as$ is surjective and its kernel is a maximal left ideal M_s. Now if $a \in \operatorname{Rad} A$ then $a \in M_s$ for every S and $s \in S$, so $as = 0$ and a annihilates every simple module. Conversely, if $a \cdot S = 0$ for every simple module S and M is a maximal left ideal then A/M is a simple module. Therefore $a \cdot (A/M) = 0$, which means $a \in M$. Hence $a \in \bigcap_{\text{maximal} M} M = \operatorname{Rad} A$.

(2) Being the intersection of left ideals, $\operatorname{Rad} A$ is also a left ideal of A. Suppose that $a \in \operatorname{Rad} A$ and $b \in A$, so $a \cdot S = 0$ for every simple S. Now $a \cdot bS \subseteq a \cdot S = 0$ so ab has the same property that a does.

(3) (a) and (b) are immediate from Lemma 3.4.1. We prove (c). Choose any composition series

$$0 = A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset A_0 = {}_A A$$

of the regular representation. Since each A_i/A_{i+1} is a simple A-module, $\operatorname{Rad} A \cdot A_i \subseteq A_{i+1}$ by part (1). Hence $(\operatorname{Rad} A)^r \cdot A \subseteq A_r$ and $(\operatorname{Rad} A)^n = 0$.

Suppose now that I is a nilpotent ideal of A, say $I^m = 0$, and let S be any simple A-module. Then

$$0 = I^m \cdot S \subseteq I^{m-1} \cdot S \subseteq \cdots \subseteq IS \subseteq S$$

is a chain of A-submodules of S that are either 0 or S since S is simple. There must be some point where $0 = I^r S \neq I^{r-1} S = S$. Then $IS = I \cdot I^{r-1} S = I^r S = 0$, so in fact that point was the very first step. This shows that $I \subseteq \operatorname{Rad} A$ by part (1). Hence $\operatorname{Rad} A$ contains every nilpotent ideal of A, so is the unique largest such ideal.

Finally (d) follows from (a) and (c): these imply that $\operatorname{Rad} A$ has the properties stated in (d); and, conversely, these conditions on an ideal U imply by (a) that $U \supseteq \operatorname{Rad} A$, and by (c) that $U \subseteq \operatorname{Rad} A$. $\square$

Working in the generality of a finite dimensional algebra A again, the radical of A allows us to give a further description of the radical and socle of a module. We present this result for finite dimensional modules, but it is in fact true without this hypothesis. We leave this stronger version to Exercise **??** at the end of this chapter.

Proposition 3.4.3. *Let A be a finite dimensional algebra over a field k, and U a finite dimensional A-module.*

(1) The following are all descriptions of $\operatorname{Rad} U$:

- *(a) the intersection of the maximal submodules of U,*
- *(b) the smallest submodule of U with semisimple quotient,*
- *(c)* $\operatorname{Rad} A \cdot U$.

(2) The following are all descriptions of $\operatorname{Soc} U$:

- *(a) the sum of the simple submodules of U,*
- *(b) the largest semisimple submodule of U,*
- *(c)* $\{u \in U \mid \operatorname{Rad} A \cdot u = 0\}$.

Proof. Under the hypothesis that U is finitely generated we have seen the equivalence of descriptions (a) and (b) in Lemma 3.4.1 and Corollary **??**. Our arguments below actually work without the hypothesis of finite generation, provided we assume the results of Exercises **??** and **??** from Chapter 1. The reader who is satisfied with a proof for finitely generated modules can assume that the equivalence of (a) and (b) has already been proved.

Let us show that the submodule $\operatorname{Rad} A \cdot U$ in (1)(c) satisfies condition (1)(b). Firstly $U/(\operatorname{Rad} A \cdot U)$ is a module for $A/\operatorname{Rad} A$, which is a semisimple algebra. Hence $U/(\operatorname{Rad} A \cdot U)$ is a semisimple module and so $\operatorname{Rad} A \cdot U$ contains the submodule of (1)(b). On the other hand if $V \subseteq U$ is a submodule for which U/V is semisimple then $\operatorname{Rad} A \cdot (U/V) = 0$ by Proposition 3.4.2, so $V \supseteq \operatorname{Rad} A \cdot U$. In particular, the submodule of (1)(b) contains $\mathrm{Rad}\, A \cdot U$. This shows that the descriptions in (1)(b) and (1)(c) are equivalent.

To show that they give the same submodule as (1)(a), observe that if V is any maximal submodule of U, then as above (since U/V is simple) $V \supseteq \mathrm{Rad}\, A \cdot U$, so the intersection of maximal submodules of U contains $\mathrm{Rad}\, A \cdot U$. The intersection of maximal submodules of the semisimple module $U/(\mathrm{Rad}\, A \cdot U)$ is zero, so this gives a containment the other way, since they all correspond to maximal submodules of U. We deduce that the intersection of maximal submodules of U equals $\mathrm{Rad}\, A \cdot U$.

For the conditions in (2), observe that $\{u \in U \mid \mathrm{Rad}\, A \cdot u = 0\}$ is the largest submodule of U annihilated by $\mathrm{Rad}\, A$. It is thus an $A/\mathrm{Rad}\, A$-module and hence is semisimple. Since every semisimple submodule of U is annihilated by $\mathrm{Rad}\, A$, it equals the largest such submodule. □

3.5 Projective covers, Nakayama's lemma and lifting of idempotents

We now develop the theory of projective covers. We first make the definition that an *essential epimorphism* is an epimorphism of modules $f : U \to V$ with the property that no proper submodule of U is mapped surjectively onto V by f. An equivalent formulation is that whenever $g : W \to U$ is a map such that fg is an epimorphism, then g is an epimorphism. One immediately asks for examples of essential epimorphisms, but it is probably more instructive to consider epimorphisms that are not essential. If $U \to V$ is any epimorphism and X is a non-zero module then the epimorphism $U \oplus X \to V$ constructed as the given map on U and zero on X can never be essential. This is because U is a submodule of $U \oplus X$ mapped surjectively onto V. Thus if $U \to V$ is essential then U can have no direct summands that are mapped to zero. One may think of an essential epimorphism as being minimal, in that no unnecessary parts of U are present.

Class: which are essential? $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}^2 \xrightarrow{(1,1)} \mathbb{Z}$, $k[x]/(x^5) \to k[x]/(x^4)$

The greatest source of essential epimorphisms is Nakayama's lemma, given here in a version for modules over non-commutative rings. Over an arbitrary ring a finiteness condition is required, and that is how we state the result here. It is a fact that, when the ring is a finite dimensional algebra over a field, the result is true for arbitrary modules without any finiteness condition.

Theorem 3.5.1 (Nakayama's Lemma). *If U is any Noetherian module, the homomorphism $U \to U/\mathrm{Rad}\, U$ is essential. Equivalently, if V is a submodule of U with the property that $V + \mathrm{Rad}\, U = U$, then $V = U$.*

Proof. Suppose V is a submodule of U. If $V \neq U$ then $V \subseteq M \subset U$ where M is a maximal submodule of U. Now $V + \mathrm{Rad}\, U \subseteq M$ and so the composite $V \to U \to U/\mathrm{Rad}\, U$ has image contained in $M/\mathrm{Rad}\, U$, which is not equal to $U/\mathrm{Rad}\, U$ since $(U/\mathrm{Rad}\, U)/(M/\mathrm{Rad}\, U) \cong U/M \neq 0$. □

When U is a module for a finite dimensional algebra it is always true that every proper submodule of U is contained in a maximal submodule, even when U is not finitely generated. This was the only point in the proof of Theorem 3.5.1 where the Noetherian hypothesis was used, and so in this situation $U \to U/\operatorname{Rad} U$ is always essential. This is shown in Exercise **??** of this chapter.

The next result is not at all difficult and could also be proved as an exercise.

Proposition 3.5.2. *(1) Suppose that $f : U \to V$ and $g : V \to W$ are two module homomorphsms. If two of f, g and gf are essential epimorphisms then so is the third.*

(2) Let $f : U \to V$ be a homomorphism of Noetherian modules. Then f is an essential epimorphism if and only if the homomorphism of radical quotients $U/\operatorname{Rad} U \to V/\operatorname{Rad} V$ is an isomorphism.

(3) Let $f_i : U_i \to V_i$ be homomorphisms of Noetherian modules, where $i = 1, \ldots, n$. The f_i are all essential epimorphisms if and only if

$$\oplus f_i : \bigoplus_i U_i \to \bigoplus_i V_i$$

is an essential epimorphism.

Proof. (1) Suppose f and g are essential epimorphisms. Then gf is an epimorphism also, and it is essential because if U_0 is a proper submodule of U then $f(U_0)$ is a proper submodule of V since f is essential, and hence $g(f(U_0))$ is a proper submodule of S since g is essential.

Next suppose f and gf are essential epimorphisms. Since $W = \operatorname{Im}(gf) \subseteq \operatorname{Im}(g)$ it follows that g is an epimorphism. If V_0 is a proper submodule of V then $f^{-1}(V_0)$ is a proper submodule of U since f is an epimorphism, and now $g(V_0) = gf(f^{-1}(V_0))$ is a proper submodule of S since gf is essential.

Suppose that g and gf are essential epimorphisms. If f were not an epimorphism then $f(U)$ would be a proper submodule of V, so $gf(U)$ would be a proper submodule of W since gf is essential. Since $gf(U) = W$ we conclude that f is an epimorphism. If U_0 is a proper submodule of U then $gf(U_0)$ is a proper submodule of W, since gf is essential, so $f(U_0)$ is a proper submodule of V since g is an epimorphism. Hence f is essential.

(2) Consider the commutative square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ U/\operatorname{Rad} U & \longrightarrow & V/\operatorname{Rad} V \end{array}$$

where the vertical homomorphisms are essential epimorphisms by Nakayama's lemma. Now if either of the horizontal arrows is an essential epimorphism then so is the other, using part (1). The bottom arrow is an essential epimorphism if and only if it is an isomorphism; for $U/\operatorname{Rad} U$ is a semisimple module and so the kernel of the map to

$V/\operatorname{Rad} V$ has a direct complement in $U/\operatorname{Rad} U$, which maps onto $V/\operatorname{Rad} V$. Thus if $U/\operatorname{Rad} U \to V/\operatorname{Rad} V$ is an essential epimorphism its kernel must be zero and hence it must be an isomorphism.

(3) The map
$$(\oplus_i U_i)/\operatorname{Rad}(\oplus_i U_i) \to (\oplus_i V_i)/\operatorname{Rad}(\oplus_i V_i)$$
induced by $\oplus f_i$ may be identified as a map
$$\bigoplus_i (U_i/\operatorname{Rad} U_i) \to \bigoplus_i (V_i/\operatorname{Rad} V_i),$$
and it is an isomorphism if and only if each map $U_i/\operatorname{Rad} U_i \to V_i/\operatorname{Rad} V_i$ is an isomorphism. These conditions hold if and only if $\oplus f_i$ is an essential epimorphism, if and only if each f_i is an essential epimorphism by part (2). □

We define a *projective cover* of a module U to be an essential epimorphism $P \to U$, where P is a projective module. Strictly speaking the projective cover is the homomorphism, but we may also refer to the module P as the projective cover of U. We are justified in calling it *the* projective cover by the second part of the following result, which says that projective covers (if they exist) are unique.

Proposition 3.5.3. *(1) Suppose that $f : P \to U$ is a projective cover of a module U and $g : Q \to U$ is an epimorphism where Q is a projective module. Then we may write $Q = Q_1 \oplus Q_2$ so that g has components $g = (g_1, 0)$ with respect to this direct sum decomposition and $g_1 : Q_1 \to U$ appears in a commutative triangle*

$$\begin{tikzcd} & Q_1 \arrow[dl, "\gamma"'] \arrow[d, "g_1"] \\ P \arrow[r, "f"] & U \end{tikzcd}$$

where γ is an isomorphism.

(2) If any exist, the projective covers of a module U are all isomorphic, by isomorphisms that commute with the essential epimorphisms.

Proof. (1) In the diagram

$$\begin{tikzcd} & Q \arrow[d, "g"] \\ P \arrow[r, "f"] & U \end{tikzcd}$$

we may lift in both directions to obtain maps $\alpha : P \to Q$ and $\beta : Q \to P$ so that the two triangles commute. Now $f\beta\alpha = g\alpha = f$ is an epimorphism, so $\beta\alpha$ is also an epimorphism since f is essential. Thus β is an epimorphism. Since P is projective β splits and $Q = Q_1 \oplus Q_2$ where $Q_2 = \ker \beta$, and β maps Q_1 isomorphically to P. Thus $g = (f\beta|_{Q_1}, 0)$ is as claimed with $\gamma = \beta|_{Q_1}$.

(2) Supposing that $f : P \to U$ and $g : Q \to U$ are both projective covers, since Q_1 is a submodule of Q that maps onto U and f is essential we deduce that $Q = Q_1$. Now $\gamma : Q \to P$ is the required isomorphism. □

Corollary 3.5.4. *If P and Q are Noetherian projective modules over a ring then $P \cong Q$ if and only if $P/\operatorname{Rad} P \cong Q/\operatorname{Rad} Q$.*

Proof. By Nakayama's lemma P and Q are the projective covers of $P/\operatorname{Rad} P$ and $Q/\operatorname{Rad} Q$. It is clear that if P and Q are isomorphic then so are $P/\operatorname{Rad} P$ and $Q/\operatorname{Rad} Q$, and conversely if these quotients are isomorphic then so are their projective covers, by uniqueness of projective covers. □

If P is a projective module for a finite dimensional algebra A then Corollary 3.5.4 says that P is determined up to isomorphism by its semisimple quotient $P/\operatorname{Rad} P$. We are going to see that if P is an indecomposable projective A-module, then its radical quotient is simple, and also that every simple A-module arises in this way. Furthermore, every indecomposable projective for a finite dimensional algebra is isomorphic to a summand of the regular representation (something that is not true in general for projective $\mathbb{Z}G$-modules, for instance). This means that it is isomorphic to a module Af for some primitive idempotent $f \in A$, and the radical quotient $P/\operatorname{Rad} P$ is isomorphic to $(A/\operatorname{Rad} A)e$ where e is a primitive idempotent of $A/\operatorname{Rad} A$ satisfying $e = f + \operatorname{Rad} A$. We will examine this kind of relationship between idempotent elements more closely.

In general if I is an ideal of a ring A and f is an idempotent of A then clearly $e = f + I$ is an idempotent of A/I, and we say that f *lifts* e. On the other hand, given an idempotent e of A/I it may or may not be possible to lift it to an idempotent of A. If, for every idempotent e in A/I, we can always find an idempotent $f \in A$ such that $e = f + I$ then we say we can *lift idempotents* from A/I to A.

We present the next results about lifting idempotents in the context of a ring with a nilpotent ideal I, but readers familiar with completions will recognize that these results extend to a situation where A is complete with respect to the I-adic topology on A.

Theorem 3.5.5. *Let I be a nilpotent ideal of a ring A and e an idempotent in A/I. Then there exists an idempotent $f \in A$ with $e = f + I$. If e is primitive, so is any lift f.*

Proof. We define idempotents $e_i \in A/I^i$ inductively such that $e_i + I^{i-1}/I^i = e_{i-1}$ for all i, starting with $e_1 = e$. Suppose that e_{i-1} is an idempotent of A/I^{i-1}. Pick any element $a \in A/I^i$ mapping onto e_{i-1}, so that $a^2 - a \in I^{i-1}/I^i$. Since $(I^{i-1})^2 \subseteq I^i$ we have $(a^2 - a)^2 = 0 \in A/I^i$. Put $e_i = 3a^2 - 2a^3$. This does map to $e_{i-1} \in A/I^{i-1}$ and we have

Class: does this mean $(e_i + I^{i-1})/I^i$ or $e_i + (I^{i-1}/I^i)$

$$\begin{aligned} e_i^2 - e_i &= (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) \\ &= -(3 - 2a)(1 + 2a)(a^2 - a)^2 \\ &= 0. \end{aligned}$$

This completes the inductive definition, and if $I^r = 0$ we put $f = e_r$.

Suppose that e is primitive and that f can be written $f = f_1 + f_2$ where f_1 and f_2 are orthogonal idempotents. Then $e = e_1 + e_2$, where $e_i = f_i + I$, is also a sum of orthogonal idempotents. Therefore one of these is zero, say, $e_1 = 0 \in A/I$. This means that $f_1^2 = f_1 \in I$. But I is nilpotent, and so contains no non-zero idempotent. □

We will very soon see that in the situation of Theorem 3.5.5, if f is primitive, so is e. It depends on the next result, which is a more elaborate version of Theorem 3.5.5.

Corollary 3.5.6. *Let I be a nilpotent ideal of a ring A and let $1 = e_1 + \cdots + e_n$ be a sum of orthogonal idempotents in A/I. Then we can write $1 = f_1 + \cdots + f_n$ in A, where the f_i are orthogonal idempotents such that $f_i + I = e_i$ for all i. If the e_i are primitive then so are the f_i.*

Proof. We proceed by induction on n, the induction starting when $n = 1$. Suppose that $n > 1$ and the result holds for smaller values of n. We will write $1 = e_1 + E$ in A/I where $E = e_2 + \cdots + e_n$ is an idempotent orthogonal to e_1. By Theorem 3.5.5 we may lift e_1 to an idempotent $f_1 \in A$. Write $F = 1 - f_1$, so that F is an idempotent that lifts E. Now F is the identity element of the ring FAF which has a nilpotent ideal FIF. The composite homomorphism $FAF \hookrightarrow A \to A/I$ has kernel $FAF \cap I$ and this equals FIF, since clearly $FAF \cap I \supseteq FIF$, and if $x \in FAF \cap I$ then $x = FxF \in FIF$, so $FAF \cap I \subseteq FIF$. Inclusion of FAF in A thus induces a monomorphism $FAF/FIF \to A/I$, and its image is $E(A/I)E$. In $E(A/I)E$ the identity element E is the sum of $n-1$ orthogonal idempotents, and this expression is the image of a similar expression for $F + FIF$ in FAF/FIF. By induction, there is a sum of orthogonal idempotents $F = f_2 + \cdots + f_n$ in FAF that lifts the expression in FAF/FIF and hence also lifts the expression for E in A/I, so we have idempotents $f_i \in A$, $i = 1, \ldots, n$ with $f_i + I = e_i$. These f_i are orthogonal: for $f_2, \ldots, f_n$ are orthogonal in FAF by induction, and if $i > 1$ then $Ff_i = f_i$ so we have $f_1 f_i = f_1 F f_i = 0$. In class, do the case $n = 2$.

The final assertion about primitivity is the last part of Theorem 3.5.5. □

Corollary 3.5.7. *Let f be an idempotent in a ring A that has a nilpotent ideal I. Then f is primitive if and only if $f + I$ is primitive.*

Proof. We have seen in Theorem 3.5.5 that if $f + I$ is primitive, then so is f. Conversely, if $f + I$ can be written $f + I = e_1 + e_2$ where the e_i are orthogonal idempotents of A/I, then by applying Corollary 3.5.6 to the ring fAf (of which f is the identity) we may write $f = g_1 + g_2$ where the g_i are orthogonal idempotents of A that lift the e_i. □

3.6 Projective modules for finite dimensional algebras

We now classify the indecomposable projective modules over a finite dimensional algebra as the projective covers of the simple modules. We first describe how these projective covers arise, and then show that they exhaust the possibilities for indecomposable projective modules. We postpone explicit examples until the next section, in which we consider group algebras.

Theorem 3.6.1. *Let A be a finite dimensional algebra over a field and S a simple A-module.*

(1) There is an indecomposable projective module P_S with $P_S/\operatorname{Rad} P_S \cong S$, of the form $P_S = Af$ where f is a primitive idempotent in A.

(2) The idempotent f has the property that $fS \neq 0$ and if T is any simple module not isomorphic to S then $fT = 0$.

(3) P_S is the projective cover of S, it is uniquely determined up to isomorphism by this property and has S as its unique simple quotient.

(4) It is also possible to find an idempotent $f_S \in A$ so that $f_S S = S$ and $f_S T = 0$ for every simple module T not isomorphic to S.

Proof. Let $e \in A/\operatorname{Rad} A$ be any primitive idempotent such that $eS \neq 0$. It is possible to find such e since we may write 1 as a sum of primitive idempotents and some term in the sum must be non-zero on S. Let f be any lift of e to A, possible by Corollary 3.5.6. Then f is primitive, $fS = eS \neq 0$ and $fT = eT = 0$ if $T \not\cong S$ since a primitive idempotent e in the semisimple ring $A/\operatorname{Rad} A$ is non-zero on a unique isomorphism class of simple modules. We define $P_S = Af$, an indecomposable projective module. Now

Class: are these true in general? $(A/\operatorname{Rad} A) \cdot e \cong S$? $eS = S$?

$$P_S/\operatorname{Rad} P_S = Af/(\operatorname{Rad} A \cdot Af) \cong (A/\operatorname{Rad} A) \cdot (f + \operatorname{Rad} A) = S,$$

the isomorphism arising because the map $Af \to (A/\operatorname{Rad} A) \cdot (f + \operatorname{Rad} A)$ defined by $af \mapsto (af + \operatorname{Rad} A)$ has kernel $(\operatorname{Rad} A) \cdot f$. The fact that P_S is the projective cover of S is a consequence of Nakayama's lemma, and the uniqueness of the projective cover was dealt with in Proposition 3.5.3. Any simple quotient of P_S is a quotient of $P_S/\operatorname{Rad} P_S$, so there is only one of these. Finally we observe that if we had written 1 as a sum of primitive central idempotents in $A/\operatorname{Rad} A$, the lift of the unique such idempotent that is non-zero on S is the desired idempotent f_S. $\square$

Class: justify these various statements. Did (3) get proved?

Theorem 3.6.2. *Let A be a finite dimensional algebra over a field k. Up to isomorphism, the indecomposable projective A-modules are exactly the modules P_S that are the projective covers of the simple modules, and $P_S \cong P_T$ if and only if $S \cong T$. Each projective P_S appears as a direct summand of the regular representation, with multiplicity equal to the multiplicity of S as a summand of $A/\operatorname{Rad} A$. As a left A-module the regular representation decomposes as*

$$A \cong \bigoplus_{\text{simple } S} (P_S)^{n_S}$$

where $n_S = \dim_k S$ if k is algebraically closed, and more generally $n_S = \dim_D S$ where $D = \operatorname{End}_A(S)$.

In what follows we will only prove that finitely generated indecomposable projective modules are isomorphic to P_S, for some simple S. For a finite dimensional algebra it is the case that this accounts for all indecomposable projective modules.

Proof. Let P be an indecomposable projective module and write

$$P/\operatorname{Rad} P \cong S_1 \oplus \cdots \oplus S_n.$$

Then $P \to S_1 \oplus \cdots \oplus S_n$ is a projective cover. Now

$$P_{S_1} \oplus \cdots \oplus P_{S_n} \to S_1 \oplus \cdots \oplus S_n$$

is also a projective cover, and by uniqueness of projective covers we have

$$P \cong P_{S_1} \oplus \cdots \oplus P_{S_n}.$$

Since P is indecomposable we have $n = 1$ and $P \cong P_{S_1}$.

Suppose that each simple A module S occurs with multiplicity n_S as a summand of the semisimple ring $A/\operatorname{Rad} A$. Both A and $\bigoplus_{\text{simple } S} P_S^{n_S}$ are the projective cover of $A/\operatorname{Rad} A$, and so they are isomorphic. We have seen in Corollary **??** that $n_S = \dim_k S$ when k is algebraically closed, and in Exercise **??** of Chapter 2 that $n_S = \dim_D S$ in general. □

Theorem 3.6.3. *Let A be a finite dimensional algebra over a field k, and U an A-module. Then U has a projective cover.*

Again, we only give a proof in the case that U is finitely generated, leaving the general case to Exercise **??** of this chapter.

Proof. Since $U/\operatorname{Rad} U$ is semisimple we may write $U/\operatorname{Rad} U = S_1 \oplus \cdots \oplus S_n$, where the S_i are simple modules. Let P_{S_i} be the projective cover of S_i and $h : P_{S_1} \oplus \cdots \oplus P_{S_n} \to U/\operatorname{Rad} U$ the projective cover of $U/\operatorname{Rad} U$. By projectivity there exists a homomorphism f such that the following diagram commutes:

$$\begin{tikzcd} & P_{S_1} \oplus \cdots \oplus P_{S_n} \arrow[dl, "f"'] \arrow[d, "h"] \\ U \arrow[r, "g"] & U/\operatorname{Rad} U \end{tikzcd}.$$

Since both g and h are essential epimorphisms, so is f by Proposition 3.5.2. Therefore f is a projective cover. □

We should really learn more from Theorem 3.6.3 than simply that U has a projective cover: the projective cover of U is the same as the projective cover of $U/\operatorname{Rad} U$.

3.7 Cartan invariants

If

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

is any composition series of a module U, the number of quotients U_i/U_{i-1} isomorphic to a given simple module S is determined independently of the choice of composition series, by the Jordan–Hölder theorem. We call this number the *(composition factor) multiplicity* of S in U.

Proposition 3.7.1. *Let S be a simple module for a finite dimensional algebra A with projective cover P_S, and let U be a finite dimensional A-module.*

(1) If T is a simple A-module then

$$\dim \operatorname{Hom}_A(P_S, T) = \begin{cases} \dim \operatorname{End}_A(S) & \text{if } S \cong T, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The multiplicity of S as a composition factor of U is

$$\dim \operatorname{Hom}_A(P_S, U)/\dim \operatorname{End}_A(S).$$

(3) If $e \in A$ is an idempotent then $\dim \operatorname{Hom}_A(Ae, U) = \dim eU$.

We remind the reader that if the ground field k is algebraically closed then $\dim \operatorname{End}_A(S) = 1$ by Schur's lemma. Thus the multiplicity of S in U is just $\dim \operatorname{Hom}_A(P_S, U)$ in this case.

Proof. (1) If $P_S \to T$ is any non-zero homomorphism, the kernel must contain $\operatorname{Rad} P_S$, being a maximal submodule of P_S. Since $P_S/\operatorname{Rad} P_S \cong S$ is simple, the kernel must be $\operatorname{Rad} P_S$ and $S \cong T$. Every homomorphism $P_S \to S$ is the composite $P_S \to P_S/\operatorname{Rad} P_S \to S$ of the quotient map and either an isomorphism of $P_S/\operatorname{Rad} P_S$ with S or the zero map. This gives an isomorphism $\operatorname{Hom}_A(P_S, S) \cong \operatorname{End}_A(S)$.

(2) Let

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

be a composition series of U. We prove the result by induction on the composition length n, the case $n = 1$ having just been established. Suppose $n > 1$ and that the multiplicity of S in U_{n-1} is $\dim \operatorname{Hom}_A(P_S, U_{n-1})/\dim \operatorname{End}_A(S)$. The exact sequence

$$0 \to U_{n-1} \to U \to U/U_{n-1} \to 0$$

gives rise to an exact sequence

$$0 \to \operatorname{Hom}_A(P_S, U_{n-1}) \to \operatorname{Hom}_A(P_S, U) \to \operatorname{Hom}_A(P_S, U/U_{n-1}) \to 0$$

by Proposition **??**, so that

$$\dim \operatorname{Hom}_A(P_S, U) = \dim \operatorname{Hom}_A(P_S, U_{n-1}) + \dim \operatorname{Hom}_A(P_S, U/U_{n-1}).$$

Dividing these dimensions by $\dim \operatorname{End}_A(S)$ gives the result, by part (1).

(3) There is an isomorphism of vector spaces $\operatorname{Hom}_A(Ae, U) \cong eU$ specified by $\phi \mapsto \phi(e)$. Note here that since $\phi(e) = \phi(ee) = e\phi(e)$ we must have $\phi(e) \in eU$. This mapping is injective since each A-module homomorphism $\phi : Ae \to U$ is determined by its value on e as $\phi(ae) = a\phi(e)$. It is surjective since the equation just written down does define a module homomorphism for each choice of $\phi(e) \in eU$. $\square$

Again in the context of a finite dimensional algebra A, we define for each pair of simple A-modules S and T the integer

$$c_{ST} = \text{the composition factor multiplicity of } S \text{ in } P_T.$$

These are called the *Cartan invariants* of A, and they form a matrix $C = (c_{ST})$ with rows and columns indexed by the isomorphism types of simple A-modules, called the *Cartan matrix* of A.

Corollary 3.7.2. *Let A be a finite dimensional algebra over a field, let S and T be simple A-modules and let e_S, e_T be idempotents so that $P_S = Ae_S$ and $P_T = Ae_T$ are projective covers of S and T. Then*

$$c_{ST} = \dim \operatorname{Hom}_A(P_S, P_T)/\dim \operatorname{End}_A(S) = \dim e_S A e_T/\dim \operatorname{End}_A(S).$$

If the ground field k is algebraically closed then $c_{ST} = \dim \operatorname{Hom}_A(P_S, P_T) = \dim e_S A e_T$.

3.8 Projective and simple modules for $S_F(n, r)$

To summarize from Theorem 3.3.3:

Corollary 3.8.1. *The isomorphism classes of simple $S_F(n, r)$-modules biject with the isomorphism classes of indecomposable projective $S_F(n, r)$-modules. These all have the form $Y^\natural = \operatorname{Hom}_{FS_r}(Y, E^{\otimes r})$ where Y is an indecomposable summand of $E^{\otimes r}$, and if $e \in S_F(n, r)$ is projection onto Y in the direct sum decomposition then $Y^\natural \cong S_F(n, r)e$ as $S_F(n, r)$-modules. The idempotent e is primitive. For idempotents e_1 and e_2 it is the case that $S_F(n, r)e_1 \cong S_F(n, r)e_2$ as $S_F(n, r)$-modules if and only if e_1 and e_2 are conjugate by a unit in $S_F(n, r)$, if and only if $e_1 E^{\otimes r} \cong e_2 E^{\otimes r}$ as FS_r-modules. The isomorphism classes of indecomposable projective $S_F(n, r)$-modules biject with the isomorphism classes of indecomposable summands of $E^{\otimes r}$.*

For each partition λ of r with at most n parts and field F we have a Young module Y^λ for FS_r, and a simple $S_F(n, r)$-module T^λ and an indecomposable projective $S_F(n, r)$-module $Q^\lambda = Y^{\lambda\natural} = S_F(n, r)e$ where e is projection onto Y^λ as a summand of $E^{\otimes r}$..

Corollary 3.8.2. *When R is a field the Cartan invariants of $S_R(n, r)$ are*

$$c_{\lambda\mu} = \dim \operatorname{Hom}_{S_R(n,r)}(Q^\lambda, Q^\mu) = \dim \operatorname{Hom}_{RS_r}(Y^\mu, Y^\lambda).$$

Corollary 3.8.3. *Assume the only summands of the M^λ are Young modules. Then the Cartan matrix of $S_F(n, r)$ is symmetric.*

Proof. We saw in Theorem 2.4.2 that the Young modules are self-dual. Now

$$\operatorname{Hom}_{FS_r}(Y^\mu, Y^\lambda) \cong \operatorname{Hom}_{FS_r}(Y^{\lambda *}, Y^{\mu *}) \cong \operatorname{Hom}_{FS_r}(Y^\lambda, Y^\mu)$$

as vector spaces □

Example 3.8.4. Cartan invariants:

$S_{\mathbb{F}_2}(2,2)$	$[2]$	$[1^2]$
$[2]$	1	1
$[1^2]$	1	2

Theorem 3.8.5. *Each simple module T^λ is absolutely simple and the projective Q^λ is absolutely indecomposable for $S_F(n,r)$.*

THE FOLLOWING ARGUMENT IS SURELY INADEQUATE, AND PROBABLY INCORRECT. THE ARGUMENT ABOUT THE GALOIS GROUP FIXING SUMMANDS SEEMS TO BE WHERE IT GOES WRONG.

Proof. These statements are equivalent to the statement that Y^λ is absolutely indecomposable for FS_r. If this were not the case there would be a finite degree field extension $K \supseteq F$ so that $Y^\lambda \otimes_F K$ is the direct sum of various submodules. The Galois group of K over F permutes these summands, which are all submodules of M^λ. Only one of them can contain S^λ which is fixed by the Galois group, so the Galois group fixes the summand containing S^λ, which is thus defined over F. It follows that there is, in fact, only one summand. □

Corollary 3.8.6. *The dimension of the simple module T^λ for $S_F(n,r)$ equals the multiplicity of Q^λ as a summand of $S_F(n,r)$, which equals the multiplicity of Y^λ as a summand of $E^{\otimes r}$ as an FS_r-module.*

We have already seen that, when $n \geq r$, $M^{[1^r]} \cong FS_r$ is a summand of $E^{\otimes r}$, and so the summands of $E^{\otimes r}$ as an $S_F(n,r)$-module are projective modules Q^λ, occurring in $E^{\otimes r}$ with multiplicities equal to the multiplicities of the Y^λ in FS_r. These Y^λ are the ones that happen to be projective, and the multiplicities are the dimensions of the corresponding simple FS_r-modules. Note that a projective Y^λ does not usually have the simple D^λ as its simple quotient. We have not proved it, but the λ for which Y^λ is projective are the partitions for which the conjugate partition λ' is p-regular.

Should this connect with result 3.3.7?

Theorem 3.8.7. *If $n \geq r$ the map $FS_r \to \mathrm{End}_{S_F(n,r)}(E^{\otimes r})$ is an isomorphism.*

Proof. We know that $E^{\otimes r} \cong M^{[1^r]\natural}$ as $S_F(n,r)$-modules, so

$$\mathrm{End}_{S_F(n,r)}(E^{\otimes r}) \cong \mathrm{End}_{S_F(n,r)}(M^{[1^r]\natural}) \cong \mathrm{End}_{FS_r}(M^{[1^r]})^{\mathrm{op}}$$

since $\natural$ is a contravariant equivalence of categories. Now $M^{[1^r]} \cong FS_r$, the regular representation, and so $\mathrm{End}_{FS_r}(M^{[1^r]}) \cong (FS_r)^{\mathrm{op}}$. This shows that $\mathrm{End}_{S_F(n,r)}(E^{\otimes r}) \cong FS_r$. To see that the map is an isomorphism, we observe first that because $M^{[1^r]}$ is a summand of $E^{\otimes r}$ and FS_r acts faithfully on it, the map is a monomorphism. By counting dimensions, it is an isomorphism. □

Corollary 3.8.8. *If $n \geq r$ then the functors denoted $\natural$ provide inverse bijections between the isomorphism classes of indecomposable summands of $E^{\otimes r}$ as a $S_F(n,r)$-module, and the indecomposable projective FS_r-modules. The multiplicity of each indecomposable $S_F(n,r)$-module as a summand of $E^{\otimes r}$ equals the multiplicity of the corresponding indecomposable as a summand of FS_r, namely the dimension of its simple quotient.*

Proof. Applying Theorem 3.3.3 to $E^{\otimes r}$ as a $S_F(n,r)$-module we have that the indecomposable summands of this module biject with the indecomposable projective FS_r-modules under $\natural$. However, these summands all have the form $Y^\natural$, where Y is a summand of the regular representation $M^{[1^r]}$. It follows that the correspondents under $\natural$ of the summands of $E^{\otimes r}$ as a $S_F(n,r)$-module are precisely the indecomposable projective FS_r-modules. □

3.9 Duality and the Schur algebra

The transpose of a linear map is familiar from elementary linear algebra as an operation on matrices. We explain its connection to bilinear forms.

Let U, V be free R-modules with bases $u_1, \ldots, u_m$, $v_1, \ldots, v_n$ giving standard bilinear forms $\langle -, - \rangle_U$ and $\langle -, - \rangle_V$ on U and V, respectively. If $\alpha : U \to V$ is a linear map, the transpose of α is the map $\bar{\alpha} : V \to U$ defined by $\langle \alpha u, v \rangle = \langle u, \bar{}v \rangle$ for all u, v.

Example 3.9.1. If $H \leq K \leq G$ are subgroups of finite index in G, the map of permutation modules $\alpha : R[H/G] \to R[K/G]$ specified by $Hg \mapsto Kg$ has transpose, with respect to the standard bilinear forms on these permutation modules, specified by $Kg \mapsto \sum_{Hx \subseteq K} Hxg$. Provided the cosets of H are ordered so that subsets of the same coset Kg are grouped together, the matrix of the transpose has the form

$$\begin{array}{cccccccc} 1 & 1 & 1 & \cdots & & & & \\ & & & & 1 & 1 & 1 & \cdots \\ & & & & & & \cdots & \end{array}$$

Proposition 3.9.2. *Let Ω, Ψ be G-sets for a finite group G and let $\alpha : R\Omega \to R\Psi$ be an RG-module homomorphism between the permutation modules. Let $\bar{\alpha} : R\Psi \to R\Omega$ be the map that is the transpose of α with respect to the standard bilinear form determined by the permutation bases of $R\Omega$ and $R\Psi$. Then $\bar{\alpha}$ is an RG-module homomorphism.*

Proof. $\bar{\alpha}$ is defined by $\langle \alpha\omega, \psi \rangle = \langle \omega, \bar{\alpha}\psi \rangle$. Now

$$\begin{aligned} \langle \omega, \bar{\alpha} g \psi \rangle &= \langle \alpha\omega, g\psi \rangle \\ &= \langle g^{-1}\alpha\omega, \psi \rangle \\ &= \langle \alpha g^{-1}\omega, \psi \rangle \\ &= \langle g^{-1}\omega, \bar{\alpha}\psi \rangle \\ &= \langle \omega, g\bar{\alpha}\psi \rangle \end{aligned}$$

for all ω, ψ and g. Because the form is non-degenerate, this shows that $\bar{\alpha}$ is an RG-module homomorphism. □

We have now defined an antiautomorphism $\alpha \mapsto \bar{\alpha}$ on $S_F(n,r)$. (It seems to depend on the particular choice of permutation basis of $E^{\otimes r}$.) If U is an $S_F(n,r)$-module, let $U^\circ = \mathrm{Hom}_R(U,R)$ with $S_F(n,r)$-action given by $(\alpha f)(u) = f(\bar{\alpha}u)$. Then U° is an $S_F(n,r)$-module.

Proposition 3.9.3. *Let U, V be $S_F(n,r)$-modules and let $\langle -,- \rangle : U \times V \to R$ be a bilinear form.*

1. *The corresponding map $U \to V^\circ$ is a map of $S_F(n,r)$-modules if and only if the form satisfies $\langle \alpha u, v\rangle = \langle u, \bar{\alpha}v\rangle$ for all u, v, α.* Class: isn't this the definition of $\bar{\alpha}$?
2. *Assuming the condition in (1) holds, if $U_1 \subseteq U$ and $V_1 \subseteq V$ are $S_F(n,r)$-submodules of U and V then $U_1^\perp \subseteq V$, $V_1^\perp \subseteq U$ are $S_F(n,r)$-submodules also.*
3. *Assuming the condition in (1) holds and that the form is non-degenerate, $U_1 \leftrightarrow U_1^\perp$ is an order-reversing binection between pure submodules of U and pure submodules of V. Furthermore, $U_1 \cong (V/U_1^\perp)^\circ$ and $U/U_1 \cong (U_1^\perp)^\circ$ as $S_F(n,r)$-modules.* Check this!

Proof. (1) The corresponding map is $u \mapsto (v \mapsto \langle u, v\rangle)$. It is a map of $S_F(n,r)$-modules if and only if αu is sent to $\alpha(v \mapsto \langle u, v\rangle)$, which equals $v \mapsto \langle u, \bar{\alpha}v\rangle$. Now $\alpha u \mapsto (v \mapsto \langle \alpha u, v\rangle)$ and these expressions are equal if and only if $\langle \alpha u, v\rangle = \langle u, \bar{\alpha}v\rangle$ for all u, v. □

Proposition 3.9.4. *When R is a field, $U \mapsto U^\circ$ is a duality on $S_F(n,r)$-modules. It interchanges injective modules and projective modules, and sends simple modules to simple modules.*

Proposition 3.9.5. *$E^{\otimes r} \cong (E^{\otimes r})^\circ$ as $S_F(n,r), FS_r$-bimodules. It follows that $E^{\otimes r}$ is injective as an $S_F(n,r)$-module, as well as being projective.*

Proof. Let $\langle -,- \rangle$ be the standard form on $E^{\otimes r}$ with respect to a permutation basis. From the definition of the antiautomorphism we have $\langle \alpha u, v\rangle = \langle u, \bar{\alpha}v\rangle$ always, which was the condition in Proposition 3.9.3. Also $\langle u\pi, v\rangle = \langle u, v\pi^{-1}\rangle$, which was the condition to get an FS_r-isomorphism. □

In general, we see that an $S_F(n,r)$-module V is an image of $E^{\otimes r}$ if and only if V° is a submodule of $(E^{\otimes r})^\circ$. As an instance of this we have the following.

Proposition 3.9.6. $\mathrm{Sym}^\lambda(E) \cong \mathrm{ST}^\lambda(E)^\circ$ *as $S_F(n,r)$-modules, and is an injective $S_F(n,r)$-module. In particular,* $\mathrm{Sym}^r(E) \cong \mathrm{ST}^r(E)^\circ$ *as $S_F(n,r)$-modules, and is injective.*

Proof. The symmetric power $\mathrm{Sym}^r(E)$ is $E^{\otimes r}/X$ where X is the span of tensors

$$(\cdots \otimes u \otimes \cdots \otimes v \otimes \cdots) - (\cdots \otimes v \otimes \cdots \otimes u \otimes \cdots)$$

Now $X = \mathrm{ST}^r(E)^{\perp}$. Hence $\mathrm{Sym}^r(E) \cong \mathrm{ST}^r(E)^{\circ}$ as $S_F(n,r)$-modules, and is injective because $ST^r(E)$ was shown to be projective. This does the case of $\lambda = [r]$ and the case of general λ can be deduced in a similar way. □

Proposition 3.9.7. *The simple $S_F(n,r)$-modules V with $eV \neq 0$ are precisely the simple $S_F(n,r)$-modules that appear as submodules of $E^{\otimes r}$.*

Should this be an exercise?

Example 3.9.8. I am not sure if this should be here: $(\mathbb{R}^4)^{\otimes 3} = 4M^{[3]} \oplus 12M^{[2,1]} \oplus 4M^{[1^3]} = 20S^{[3]} \oplus 21S^{[2,1]} \oplus 4S^{[1^3]}$.

Example when $r = 2$?

Chapter 4

Polynomial representations of $GL_n(F)$

We regard $GL_n(F)$ as the group of $n \times n$ invertible matrices with entries in F. For each i, j with $1 \leq i, j \leq n$ and $n \times n$ matrix $g \in M_n(F)$ let $c_{i,j}(g) \in F$ be the (i, j)-entry of g. We let $A_F(n)$ be the polynomial ring $F[c_{i,j} \mid 1 \leq i, j \leq n]$. When F is infinite we may regard $A_F(n)$ as the algebra of polynomial functions on $GL_n(F)$, or on $M_n(F)$. It is a direct sum $A_F(n) = \bigoplus_{n \geq 0} A_F(n, r)$ where $A_F(n, r)$ is the space of homogeneous polynomials of degree r. A standard calculation shows that $A_F(n, r)$ has dimension $\binom{n+r-1}{r}$.

Given a representation $\rho : GL_n(F) \to GL_d(F)$, let $\rho_{i,j} : GL_n(F) \to F$ be the function that extracts the (i, j)-entry of the matrix $\rho(g)$, $g \in GL_n(F)$. When F is infinite we say that ρ is a *polynomial representation* if $\rho_{i,j} \in A_F(n)$ for all $1 \leq i, j \leq d$, and it is *polynomial of degree* r if $\rho_{i,j} \in A_F(n, r)$ for all $1 \leq i, j \leq d$. The matrix with entries $\rho_{i,j}$ is called the *invariant matrix* of the representation. The notions of being a polynomial representation, and being polynomial of degree r, are independent of choice of basis.

Is there any difference between this matrix and ρ?

Example 4.0.1.

1. $\rho_1(g) = (1)$ for all g in $GL_n(F)$ is polynomial of degree 0;
2. $\rho_{\det}(g) = (\det g)$ for all g in $GL_n(F)$ is polynomial of degree n;
3. $\rho(g) = g$ for all g in $GL_n(F)$ is polynomial of degree 1. It is the natural representation on the space E of dimension n.
4. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\rho(g) = g \otimes g = \begin{bmatrix} aa & ab & ba & bb \\ ac & ad & bc & bd \\ ca & cb & da & db \\ cc & cd & dc & dd \end{bmatrix}$ for all g in $GL_2(F)$ is a polynomial representation of degree 2. It is the representation on $E^{\otimes 2}$.
5. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\rho(g) = \begin{bmatrix} ad + bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{bmatrix}$ for all g in $GL_2(F)$ is a polynomial

representation of degree 2. It is the representation on $\mathrm{ST}^2(E)$ using the basis $e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_1, e_2 \otimes e_2$.

6. $\rho(g) = ((\det g)^{-1})$ is not a polynomial representation.

7. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\rho(g) = \begin{bmatrix} 1 & 1 - ad + bc \\ 0 & ad - bc \end{bmatrix}$ is a polynomial representation that is not homogeneous. We check that if $g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ then

$$\begin{bmatrix} 1 & 1-\det g \\ 0 & \det g \end{bmatrix} \begin{bmatrix} 1 & 1-\det g' \\ 0 & \det g' \end{bmatrix} = \begin{bmatrix} 1 & 1-\det g \det g' \\ 0 & \det g \det g' \end{bmatrix} = \begin{bmatrix} 1 & 1-\det gg' \\ 0 & \det gg' \end{bmatrix}$$

so that this is indeed a representation.

We say that a function $T : GL_n(F) \to M_d(F)$ is *invariant* (perhaps *multiplicative* would be better?) if $T(BC) = T(B)T(C)$ always holds. Let $T_{i,j} : GL_n(F) \to F$ be the function that extracts the (i,j)-entry of the matrix obtained by applying T.

Example 4.0.2. We compute the invariant matrix and the $\Delta(\rho_{i,j})$ for the representation ρ in example 7. The invariant matrix is

$$\begin{bmatrix} 1 & 1 - c_{11}c_{22} + c_{12}c_{21} \\ 0 & c_{11}c_{22} - c_{12}c_{21} \end{bmatrix}$$

Notices that this equals

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -c_{11}c_{22} + c_{12}c_{21} \\ 0 & c_{11}c_{22} - c_{12}c_{21} \end{bmatrix}$$

and both of the matrices in this sum are multiplicative on $GL_2(F)$. Evaluated on the identity matrix they give

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

These matrices are orthogonal idempotents, with images $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We compute

$$\begin{bmatrix} 1 & 1-\det g \\ 0 & \det g \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1-\det g \\ 0 & \det g \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \det g \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so that the images of these idempotents are submodules for the action of $GL_2(F)$. The representation is seen to decompose as $F \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus F \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cong F \oplus \det$.

The polynomial functions $A_F(n)$ are an algebra under pointwise multiplication. There is also a coproduct $\Delta : A_F(n) \to A_F(n) \otimes A_F(n)$ defined on the algebra generators $c_{i,j}$ as $\Delta(c_{i,j}) = \sum_{k=1}^{n} c_{i,k} \otimes c_{k,j}$, so as to be an algebra homomorphism. Thus if B and C are $n \times n$ matrices and $f \in A_F(n)$ we have $\Delta(f)(B,C) = f(BC)$, where we interpret elements of $A_F(n) \otimes A_F(n)$ as functions $GL_n(F) \times GL_n(F) \to F$ via $(f \otimes g)(B,C) = f(B)g(C)$. There is also a mapping $\epsilon : A_F(n) \to F$ given by evaluation at the identity matrix: $\epsilon(c) = c(1)$.

Example 4.0.3. To illustrate how it works, we compute the effect of Δ on one of the matrix entries in the last example:

$$\Delta(\rho_{1,2}) = \text{Fill in calculation.}$$

Lemma 4.0.4. *1. $\Delta(A_F(n,r)) \subseteq A_F(n,r) \otimes A_F(n,r)$.*

2. $A_F(n)$ and $A_F(n,r)$ are coassociative coalgebras with coproduct Δ and counit ϵ.

3. Let $T : GL_n(F) \to M_d(F)$ be a mapping whose coordinate functions $T_{i,j}$ are polynomial. Then T is multiplicative if and only if $T_{i,j}(BC) = \Delta(T_{i,j})(B,C)$ always holds.

Notice that, unlike the coproduct, the product on $A_F(n)$ does not preserve degree of polynomials. In part 3, the evaluation of of $\Delta(T)$ on the pair (B,C) comes about via the homomorphism from $A_F(n) \otimes A_F(n)$ to functions $GL_n(F) \times GL_n(F) \to F$. Given $f, f' \in A_F(n)$ this homomorphism sends $f \otimes f'$ to the function $(g,h) \mapsto f(g)f'(h)$.

Proof. These come from direct verifications using the definition of Δ. We verify the coassociative law $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ by checking it on the generators $c_{i,j}$ and using the fact that Δ is an algebra homomorphism on $A_F(n)$. Thus

$$(1 \otimes \Delta)\Delta(c_{i,j}) = (1 \otimes \Delta)\sum_{k=1}^{n}(c_{i,k} \otimes c_{k,j}) = \sum_{k=1}^{n}(c_{i,k} \otimes \sum_{q=1}^{n}(c_{k,q} \otimes c_{q,j}))$$

and this takes the same value on triples of matrices as $(\Delta \otimes 1)\Delta(c_{i,j})$ because of associativity of matrix multiplication. ☐ Verify other laws.

Theorem 4.0.5 (Schur)**.** *Let F be an infinite field and let $\rho : GL_n(F) \to GL(V)$ be a polynomial representation of $GL_n(F)$. Then $V = \bigoplus_{k \geq 0} V_k$ as representations of $GL_n(F)$ where V_k is polynomial of degree k, and is characterized as the unique largest submodule of V with this property.*

The main conceptual idea behind the proof of this result is that the coalgebra $A(n)$ is the direct sum of the coalgebras $A(n,r)$, so that the dual $A(n)^*$ is the direct sum of algebras $A(n,r)^*$, whose identity elements are a set of central orthogonal idempotents. Identifying the $A(n,r)^*$-modules as polynomial representations of degree r gives the required decomposition. We have not yet made this identification, and present the argument in a more elementary way.

Proof. Let t be an indeterminate and write $\rho(tg) = \rho_0(g) + t\rho_1(g) + \cdots + t^r\rho_r(g)$ for some r. Thus the coordinate functions $(\rho_k)_{i,j}$ lie in $A(n,k)$ for each k. We claim that each ρ_k is multiplicative. For this we show that $\Delta((\rho_k)_{i,j})(g,h) = (\rho_k)_{i,j}(gh)$ always holds and apply Lemma 4.0.4.

Putting $t = 1$ we have

$$\rho = \rho_0 + \rho_1 + \cdots + \rho_r$$

as matrices of functions on $GL_n(F)$, and in each coordinate we have

$$\rho_{i,j} = (\rho_0)_{i,j} + (\rho_1)_{i,j} + \cdots + (\rho_r)_{i,j}.$$

Applying Δ we have

$$\Delta(\rho_{i,j}) = \Delta((\rho_0)_{i,j}) + \Delta((\rho_1)_{i,j}) + \cdots + \Delta((\rho_r)_{i,j}).$$

Now ρ is multiplicative, so

$$\begin{aligned}\rho_{i,j}(tgh) &= \Delta\rho_{i,j}(tg,h)\\ &= \Delta(\rho_0)_{i,j}(tg,h) + \Delta(\rho_1)_{i,j}(tg,h) + \cdots + \Delta(\rho_r)_{i,j}(tg,h)\\ &= \Delta(\rho_0)_{i,j}(g,h) + t\Delta(\rho_1)_{i,j}(g,h) + \cdots + t^r\Delta(\rho_r)_{i,j}(g,h)\end{aligned}$$

the last equality arising because the functions that evaluate on tg in $\Delta(\rho_k)_{i,j}(tg,h) \in A_F(n,k) \otimes A_F(n,k)$ are homogeneous of degree k. From the definition of the ρ_k we also have

$$\rho_{i,j}(tgh) = (\rho_0)_{i,j}(gh) + t(\rho_1)_{i,j}(gh) + \cdots + t^r(\rho_r)_{i,j}(gh).$$

Comparing coefficients of t^k in the two last expressions we obtain

$$\Delta((\rho_k)_{i,j})(g,h) = (\rho_k)_{i,j}(gh),$$

as required. We deduce that ρ_k is multiplicative.

We now see that the matrices $e_k := \rho_k(1)$ are idempotent, because $\rho_k(1)^2 = \rho_k(1^2) = \rho_k(1)$. Also

$$\rho(1) = \rho_0(1) + \rho_1(1) + \cdots + \rho_r(1) = e_0 + e_1 + \cdots e_r$$

shows that these idempotents sum to the identity endomorphism of V. This implies that V is the sum of the subspaces e_kV. For every g we have

$$\begin{aligned}\rho(t1g) &= \rho(t1)\rho(g)\\ &= (e_0 + te_1 + \cdots + t^r e_r)\rho(g)\\ &= e_0\rho(g) + te_1\rho(g) + \cdots + t^r e_r\rho(g)\\ &= \rho_0(g) + t\rho_1(g) + \cdots + t^r\rho_r(g)\\ &= \rho(gt1)\\ &= \cdots = \rho(g)e_0 + t\rho(g)e_1 + \cdots + t^r\rho(g)e_r.\end{aligned}$$

Equating coefficients of powers of t we find that $e_k\rho(g) = \rho(g)e_k = \rho_k(g)$, so the idempotents e_k commute with the matrices $\rho(g)$. Thus each subspace e_kV is preserved by the action of $GL_n(F)$, and an element g acts on a vector $v = e_kv \in e_kV$ via the matrix $\rho(g)e_k = \rho_k(g)$, a matrix whose entries are homogeneous polynomials of degree k. It follows that $e_kV \cap \sum_{i\neq k} e_iV = 0$ since $\rho(g)$ acts on this space by a matrix whose entries are simultaneously polynomial of degree k, and also polynomial with no terms of degree k. We have now shown that $V = \bigoplus_{k\geq 0} V_k$ where $V_k = e_kV$, as required. □

Our next goal is to show that, when F is infinite, the algebra homomorphism $FGL_n(F) \to S_F(n,r)$ is surjective, and that the representations of $GL_n(F)$ that arise from representations of $S_F(n,r)$ are precisely the polynomial representations of degree r. To do this we will factorize the homomorphism as $FGL_n(F) \to S_F^{\text{new}}(n,r) \to S_F(n,r)$ where $S_F^{\text{new}}(n,r)$ is an algebra to be defined, that we will show is isomorphic to $S_F(n,r)$.

We define $S_F^{\text{new}}(n,r) := A_F(n,r)^*$, and this is an algebra, with multiplication and unit defined as the dual of the comultiplication and counit of the coalgebra $A_F(n,r)$. We have maps

$$GL_n(F) \times A_F(n,r) \to F$$

and

$$M_n(F) \times A_F(n,r) \to F$$

given by evaluation $(g,c) \mapsto c(g) = e_g(c)$. Thus $e_g \in S_F^{\text{new}}(n,r) := A_F(n,r)^*$. Write $\Gamma := GL_n(F)$ and let F^Γ be the algebra of maps $\Gamma \to F$. Any $\kappa \in F^\Gamma$ gives rise to a linear map $F\Gamma \to F$, so that we have a bilinear form Change Γ back to $GL_n(F)$.

$$FGL_n(F) \times A_F(n,r) \to F$$

and the mapping e is defined so that the bilinear form is the composite

$$FGL_n(F) \times A_F(n,r) \xrightarrow{e\times 1} S_F^{\text{new}}(n,r) \times A_F(n,r) \to F$$

where the second map is the natural perfect pairing.

Proposition 4.0.6. *1. Given any $\phi, \theta \in A_F(n,r)^*$ their product $\phi\cdot\theta$ is given on $c \in A_F(n,r)$ by $(\phi\cdot\theta)(c) = (\phi\otimes\theta)\Delta(c)$.*

2. $e_{g_1}\cdot e_{g_2} = e_{g_1g_2}$.

3. e extends to an algebra homomorphism $e : FGL_n(F) \to S_F^{new}(n,r)$, written $u \mapsto e_u$, where if $u = \sum_{g\in\Gamma} a_g g$ then $e_u = \sum_{g\in\Gamma} a_g e_g$.

Proof. 1. The multiplication on $A_F(n,r)^*$ is the map $\mu : A_F(n,r)^* \otimes_F A_F(n,r)^* \to A_F(n,r)^*$ dual to $\Delta : A_F(n,r) \to A_F(n,r) \otimes_F A_F(n,r)$ with respect to the natural pairings between these spaces. This means that $\mu(\phi\otimes\theta)(c) = (\phi\otimes\theta)\Delta(c)$.

2. We have $(e_{g_1}\cdot e_{g_2})(c) = (e_{g_1}\otimes e_{g_2})\Delta(c)$ by part 1, and this equals $c(g_1g_2)$ from the definition of Δ. From the definition of the evaluation map, $c(g_1g_2) = e_{g_1g_2}(c)$ and Explain more.

this completes the proof.

3. is immediate from 2, since the only thing at issue is the multiplicativity of the map. $\square$

Proposition 4.0.7. *Let F be an infinite field.*

1. *The algebra homomorphism $e : FGL_n(F) \to S_F^{new}(n,r)$ is surjective.*

2. *Let $Y = \operatorname{Ker} e$ and $f \in F^\Gamma$. Then $f \in A_F(n,r)$ if and only if $f(Y) = 0$.*

Which implication is easy?

Proof. 1. If $e(FGL_n(F)) \neq S_F^{\text{new}}(n,r)$ then it is a proper linear subspace, and there is a nonzero polynomial function $c \in A_F(n,r)$ perpendicular to $\operatorname{Im}(e)$ under the pairing. Since $\operatorname{Im}(e)$ is the span of all e_g this means that $e_g(c) = c(g) = 0$ for all $g \in GL_F(n,r)$. Because F is infinite, a polynomial c that is zero everywhere in its domain, when regarded as a function, must be zero. Thus no such c can exist and e is surjective.

Explain that $c(e_g) = c(g)$.

2. To prove the implication in the direction left to right, let $u \in \operatorname{Ker} e$ and let $f \in A_F(n,r)$. Then $e_u(f) = f(u) = 0$, so that $f(Y) = 0$. For the other direction, suppose that $f(Y) = 0$. Since $S_F^{\text{new}}(n,r) \cong F\Gamma/Y$, f defines a function $f_1 : S_F^{\text{new}}(n,r) \to F$ given by $f_1(e_g) = f(g)$. Since the pairing $S_F^{\text{new}}(n,r) \times A_F(n,r) \to F$ is perfect, it has the form $f_1(\phi) = \phi(c)$ for some $C \in A_F(n,r)$. Now c and f take the same values on Γ, because if $g \in \Gamma$ then $f(g) = f_1(e_g) = e_g(c) = c(g)$. Hence $f = c$. $\square$

Corollary 4.0.8. *Let F be an infinite field.*

1. *Let $\sigma : S_F^{new}(n,r) \to \operatorname{End}_F(V)$ be a representation of $S_F^{new}(n,r)$. Then the composite $\rho : FGL_n(F) \xrightarrow{e} S_F^{new}(n,r) \xrightarrow{\sigma} \operatorname{End}_F(V)$ is a polynomial representation of degree r.*

Class: how many lines do you think it would take to prove this?

2. *Every polynomial representation of $GL_n(F)$ of degree r is isomorphic to a representation $FGL_n(F) \xrightarrow{e} S_F^{new}(n,r) \xrightarrow{\sigma} \operatorname{End}_F(V)$ for some σ.*

In this sense, the representations of $S_F^{new}(n,r)$ are the same as the polynomial representations of $GL_n(F)$ of degree r.

Proof. 1. For each pair (i,j) the function $\rho_{i,j}$ vanishes on the kernel Y of e and so is polynomial of degree r by Proposition 4.0.7.

2. Suppose that V is a polynomial representation of $GL_n(F)$ of degree r. Then by Proposition 4.0.7 again we have $\rho_{i,j}(Y) = 0$ for all i,j, so ρ arises as a representation of $S_F^{\text{new}}(n,r)$ as claimed. $\square$

Corollary 4.0.9. *Let F be an infinite field. The representation*

$$FGL_n(F) \to S_F(n,r) = \operatorname{End}_{FS_r}(E^{\otimes r})$$

factors as

$$FGL_n(F) \xrightarrow{e} S_F^{new}(n,r) \to S_F(n,r).$$

Class: Is the map $S^{\text{new}} \to S$ uniquely specified by this, or not?

Proof. The representation of $GL_n(F)$ on $E^{\otimes r}$ is given by the r-fold tensor product of the natural representation on E, and so is polynomial of degree r. $\square$

It was shown by Benson and Doty in [1] that the ring homomorphism $FGL_n(F) \to \mathrm{End}_{FS_r}(E^{\otimes r})$ is surjective provided that the number of elements of F is strictly greater than r.

4.0.1 Multi-indices

Our next goal is to prove that $S_F^{\mathrm{new}}(n,r)$ is isomorphic to $S_F(n,r)$. Before doing this we describe some properties of multi-indices. A *multi-index* of length r on n symbols is a list $(i_1, i_2, \ldots, i_r)$ of length r, where each i_j can be any of the n symbols. Let $I(n,r)$ be the set of such multi-indices. There is an action of the symmetric group S_r from the right on $I(n,r)$ given by permuting the positions of symbols. For example,

$$(1,1,4,3)(1,2,3) = (1,1,4,3)(1,2)(1,3) = (4,1,1,3).$$

When the set of n symbols is $\{1,\ldots,n\}$ each multi-index determines a *weak composition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ of r where λ_i is the number of occurrences of i in the list, that is, a list of non-negative integers that sum to r. We say that λ is the *content* of the multi-index. For example, the multi-index $(1,1,4,3)$ has content $\lambda = (2,0,1,1)$.

Lemma 4.0.10. *Let E be a vector space of dimension n. There are bijections*

$$\begin{aligned} I(n,r) &\leftrightarrow \textit{basis vectors for } E^{\otimes r} \\ I(n,r)/S_r &\leftrightarrow \textit{certain summands } M^\lambda \textit{ of } E^{\otimes r} \textit{ as an } FS_r\textit{-module} \\ (I(n,r)\times I(n,r))/S_r &\leftrightarrow \textit{monomials of degree } r \textit{ in the } c_{i,j},\ 1 \le i,j \le n \end{aligned}$$

For example, the multi-index $\mathbf{i} = (1,1,4,3)$ corresponds to the basis vector $e_{\mathbf{i}} = e_1 \otimes e_1 \otimes e_4 \otimes e_3$ of $E^{\otimes 4}$, and its content $\lambda = (2,0,1,1)$ corresponds to the summand $M^{(2,0,1,1)}$ spanned by $e_1 \otimes e_1 \otimes e_4 \otimes e_3$, $e_1 \otimes e_1 \otimes e_3 \otimes e_4$, $e_1 \otimes e_4 \otimes e_1 \otimes e_3$ and three other basic tensors. The pair of multi-indices $(1,1,4,3),(2,1,2,3))$ determines the monomial $c_{1,2}c_{1,1}c_{4,2}c_{3,3}$ and, since this equals $c_{4,2}c_{1,1}c_{3,3}c_{1,2}$ (for example) it is also determined by the pair of multi-indices $((4,1,3,1),(2,1,3,2))$.

The algebra $S_F^{\mathrm{new}}(n,r)$ has a basis dual to the $c_{\mathbf{i},\mathbf{j}} := c_{i_1,j_1} \cdot c_{i_2,j_2} \cdots c_{i_r,j_r}$, denoted $\xi_{\mathbf{i},\mathbf{j}}$. We have $\xi_{\mathbf{i},\mathbf{j}} = \xi_{\mathbf{a},\mathbf{b}}$ if and only if there exists $\pi \in S_r$ with $\mathbf{i} = \mathbf{a}\pi$ and $\mathbf{j} = \mathbf{b}\pi$.

Proposition 4.0.11. *The action of $S_F^{new}(n,r)$ on $E^{\otimes r}$ is given by*

1. $\xi(e_{\mathbf{i}}) = \sum_{\mathbf{k}} \xi(c_{\mathbf{k},\mathbf{i}}) e_{\mathbf{k}}$ *for all* $\xi \in S_F^{new}(n,r)$.

2. $$\xi_{\mathbf{a},\mathbf{b}}(e_{\mathbf{b}}) = \sum_{\mathbf{k} \in \mathbf{a}\cdot \mathrm{Stab}_{S_r}(\mathbf{b})} e_{\mathbf{k}}$$

   and $\xi_{\mathbf{a},\mathbf{b}}(e_{\mathbf{i}}) = 0$ *if* $\mathbf{i} \notin \mathbf{b}S_r$.

We do not need the hypothesis that F be an infinite field in the next result.

Theorem 4.0.12. *The algebra homomorphism $S_F^{new}(n,r) \to S_F(n,r)$ is an isomorphism.*

Proof. We have seen that $S_F(n,r)$ has a basis acting on $E^{\otimes r}$ in exactly the same way as the $\xi_{\mathbf{a},\mathbf{b}}$. $\square$

I think this refers to result 3.2.2 for this basis. Make this explicit.

Theorem 4.0.13. *Let $\rho : FGL_n(F) \to S_F(n,r)$ be the homomorphism that expresses $E^{\otimes r}$ as a representation of $GL_n(F)$ The representations of $GL_n(F)$ obtained from representations of $S_F(n,r)$ via ρ are precisely the polynomial representations of $GL_n(F)$ of degree r.*

Corollary 4.0.14. *Let F be a field of characteristic zero. The polynomial representations of $GL_n(F)$ are semisimple. The simple polynomial representations of degree r are parametrized by the partitions of r with at most n parts. Thus, when $n \geq r$, the simple polynomial representations of degree r are parametrized by the partitions of r.*

Corollary 4.0.15. *Over an infinite field F, the number of simple polynomial representations of $GL_n(F)$ of degree r is at least the number of partitions of r with $\leq n$ parts. The representations $ST^\lambda(E)$ are projective in the category of polynomial representations, and every indecomposable projective is a summand of one of these. The $\mathrm{Sym}^\lambda(E)$ are injective in this category and every indecomposable injective is a summand of one of these. The space $E^{\otimes r}$ is both projective and injective and contains all indecomposable projectives as submodules.*

4.1 Weights and Characters

Let $T \leq GL_n(F)$ be the subgroup of diagonal matrices $\tau = \mathrm{diag}(t_1, \ldots, t_n)$. The *character group* of T is the set $X(T)$ of algebraic group homomorphisms $T \to F^\times$.

Proposition 4.1.1. *The elements of $X(T)$ have the form χ_λ where $\lambda \in \mathbb{Z}^n$ and $\chi_\lambda(\mathrm{diag}(t_1, \ldots, t_n)) = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$.*

There is a binary operation on $X(T)$, written additively, given by multiplication within $F^\times$: if $\chi_1, \chi_2 \in X(T)$ then $(\chi_1 + \chi_2)(t) := \chi_1(t)\chi_2(t)$.

Proposition 4.1.2. *We have $\chi_{\lambda_1} + \chi_{\lambda_2} = \chi_{\lambda_1+\lambda_2}$, so that $X(T) \cong \mathbb{Z}^n$ as abelian groups.*

As before, let $I(n,r)$ denote the set of multi-indices of length r on n symbols, so that $I(n,r)/S_r$ is in bijection with the set $\Lambda(n,r)$ of (weak) compositions of r with at most n parts. We identify $\Lambda(n,r) = I(n,r)/S_r$ and call the elements *weights*. Each $\lambda \in \Lambda(n,r)$ determines an element χ_λ of $X(T)$, called a *character*.

The Weyl group $W \cong S_n$ of $GL_n(F)$ acts on $I(n,r)$ from the left as

$$w(i_1, \ldots, i_n) = (w(i_1), \ldots, w(i_n))$$

and this action commutes with the action of S_r from the right, so the action of W passes to an action on $\Lambda(n,r)$.

Example 4.1.3. The multi-index $(2,2,1,4,2,1)$ determines a composition $(2,3,0,1)$ which is the the same for all multi-indices in the right orbit of S_6. Letting S_4 act from the left, it is in the same orbit as $(1,1,2,3,1,2)$ via the permutation $(1,2)(3,4)$. This multi-index determines the partition $(3,2,1,0)$, which is the image of $(2,3,0,1)$ under this permutation.

We call the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ *dominant* if $\lambda_1 \geq \cdots \geq \lambda_n$ which is the same as saying that λ is a partition. Each W-orbit on the weights contains a unique dominant weight. We write $\Lambda(n,r)^+$ for the set of dominant weights of $GL_n(F)$ of degree r.

Let V be a polynomial representation of $GL_n(F)$ and let $\lambda \in \Lambda(n)$ be a weight. We define the *weight space*

$$V^\lambda = \{v \in V \mid \tau v = \chi_\lambda(\tau) \cdot v \text{ for all } \tau \in T\}.$$

We say that λ is a weight of V if $V^\lambda \neq 0$. Since these weight spaces are common eigenspaces and V has finite dimension, V has only finitely many weights.

Example 4.1.4. When V is the trivial module for $GL_n(F)$, every element of T acts via $\lambda = (0,0,\ldots) = 0$ so $V = V^{(0,0,\ldots)}$.

Example 4.1.5. When V is the natural representation with basis $e_1, \ldots, e_n$ and $\lambda = (0,0,\ldots,0,1,0\ldots)$ with 1 in position i then $V^\lambda = Fe_i$. These are the only weight spaces, and V is their direct sum. We see that the multi-index (i) – merely an index in this case – corresponds to the basis vector e_i and determines the composition λ. We also see that the weights are in a single orbit under the action of W. Class: how many weights are there?

Example 4.1.6. The exterior power $V = \bigwedge^r(E)$ is a vector space with basis the symbols $e_{i_1} \wedge \cdots \wedge e_{i_r}$ where $i_1 < \cdots < i_r$. A wedge of r vectors satisfies linearity in each variable and skew-symmetry. We see that each wedge $e_{i_1} \wedge \cdots \wedge e_{i_r}$ spans a weight space of V with weight λ, where $\lambda_j = 1$ if $j \in \{i_1, \ldots, i_r\}$ and otherwise $\lambda_j = 0$. These weights form a single W-orbit, and the dominant weight in this orbit is $[1^r]$.

Proposition 4.1.7. *Let V be a polynomial representation of degree r.*

1. *If λ is a weight of V then λ has degree r.*
2. *If $\pi \in W$ then $V^{\pi\lambda} = \pi V^\lambda$ and as vector spaces V^λ and $V^{\pi\lambda}$ have the same dimension.*

Proof. 1. The diagonal subgroup T acts via matrices all of whose entries are polynomial of degree r. On V^λ each $\tau \in T$ acts as a scalar that is polynomial of degree r in the entries and so λ must be a composition of r.

2. Let $v \in V^\lambda$ and $\tau \in T$. Then $\tau\pi v = \pi(\pi^{-1}\tau\pi)v$ and if $\tau = \mathrm{diag}(t_1, \ldots, t_n)$ then $\pi^{-1}\tau\pi = \mathrm{diag}(t_{\pi 1}, \ldots, t_{\pi n}$. This acts on V^λ as $t_{\pi 1}^{\lambda_1} \cdots t_{\pi n}^{\lambda_n} = t_1^{\lambda_{\pi^{-1}1}} \cdots t_n^{\lambda_{\pi^{-1}n}} = \chi_{\pi\lambda}(\tau)$. This shows that $V^{\pi\lambda}$ is preserved by τ, and that $V^{\pi\lambda} = \pi V^\lambda$. $\square$

Recall the elements $\xi_{\mathbf{i},\mathbf{j}} \in S_F(n,r)$ dual to the $c_{\mathbf{i},\mathbf{j}}$. For each $\mathbf{i},\mathbf{j} \in I(n,r)$ we have $\xi_{\mathbf{i},\mathbf{i}} = \xi_{\mathbf{j},\mathbf{j}}$ if and only if $\mathbf{j} = \mathbf{i}\pi$ for some $\pi \in S_r$, and so if $\lambda = \mathbf{i} \cdot S_r \in \Lambda(n,r)$ it is well-defined to write ξ_λ instead of $\xi_{\mathbf{i},\mathbf{i}}$ if $\mathbf{i} \in \lambda$.

As an example, the multi-index $\{\mathbf{i} = (2,1,5,4,4)$ determines the composition $\lambda = (1,1,0,2,1)$ and the summand $M^{(1,1,0,2,1)}$ of $E^{\otimes r}$, which is the span of vectors such as $e_2 \otimes e_1 \otimes e_5 \otimes e_4 \otimes e_4 = e_{\mathbf{i}}$ and the other tensors with the same content. According to Proposition 4.0.11, $\xi_{\mathbf{i},\mathbf{i}} e_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathbf{i}} \mathrm{Stab}(\mathbf{i}) e_{\mathbf{j}} = e_{\mathbf{i}}$, from which it follows that $\xi_{\mathbf{i},\mathbf{i}} = \xi_\lambda$ is the identity on M^λ, and the same proposition shows that $\xi_{\mathbf{i},\mathbf{i}} e_{\mathbf{j}} = 0$ if $\mathbf{j}$ is not in the same S_r-orbit as $e_{\mathbf{i}}$. This proves the first part of the next proposition.

The summation index may be wrong

Proposition 4.1.8. *1. Let $\lambda \in I(n,r)$. Then ξ_λ is projection onto the summand M^λ of $E^{\otimes r}$.*

2. In the Schur algebra, $1 = \sum_{\alpha\in\Lambda(n,r)} \xi_\alpha$ is a sum of orthognal idempotents.

3. If V is any polynomial representation of degree r then $V = \bigoplus_{\alpha\in\Lambda(n,r)} \xi_\alpha V$.

4. For each $\tau = \mathrm{diag}(t_1,\ldots,t_n) \in T$ its image in $S_F(n,r)$ is

$$e_\tau = \sum_{\alpha\in\Lambda(n,r)} t_1^{\alpha_1} \cdots t_n^{\alpha_n} \xi_\alpha.$$

Proof. 1. follows from the argument just before the proposition.

2. follows from 1. because the distinct ξ_α are orthogonal, and their sum acts as the identity on $E^{\otimes r}$.

3. This decomposition follows from 2.

4. From the definition of the action, $e_\tau \in S_F(n,r)$ acts on $E^{\otimes r}$ the way τ does. We see that τ acts on each basis element $e_{\mathbf{i}}$ of $M^\alpha = \xi_\alpha E^{\otimes r}$ as scalar multiplication by $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, so e_τ acts on $\xi_\alpha E^{\otimes r}$ in this way. The result follows. We may also evaluate each side of the formula on $c_{\mathbf{i},\mathbf{j}}$ to obtain this result. □

Theorem 4.1.9. *Let V be a polynomial representation of degree r. Then*

$$V = \bigoplus_{\alpha\in\Lambda(n,r)} V^\alpha$$

is the direct sum of its weight spaces.

Proof. Because τ acts on $\xi_\alpha V$ as multiplication by $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ we have $\xi_\alpha V \subseteq V^\alpha$. However, $V^\alpha \cap \sum_{\beta\neq\alpha} V^\beta = 0$ and this forces the result. □

Corollary 4.1.10. *The diagonal subgroup T acts semisimply on every polynomial representation of $GL_n(F)$.*

Proof. We see that any polynomial representation V is the direct sum of spaces on which elements of T act as scalar multiplication, each of which is a direct sum of one-dimensional subspaces preserved under the action of T. □

Proposition 4.1.11. *1. Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact sequence of $S_F(n,r)$-modules. Then for all weights α we have that $0 \to V_1^\alpha \to V^\alpha \to V_2^\alpha \to 0$ is exact.*

2. Let V and W be polynomial representations of $GL_n(F)$ of degrees r and s. Then $V \otimes_F W$ is a polynomial representation of degree $r+s$ and if $\gamma \in \Lambda(n, r+s)$ then

$$(V \otimes W)^\gamma = \bigoplus_{\alpha+\beta=\gamma} V^\alpha \otimes_F W^\beta.$$

3. If $L \supseteq F$ is a field extension and V is an $S_F(n,r)$-module then the $S_L(n,r)$-module $L \otimes_F V$ has $(L \otimes_F V)^\alpha = L \otimes_F V^\alpha$.

Proof. 1. arises because of the semisimplicity of the action of T and the fact that if $V \to W$ is a homomorphism of $S_F(n,r)$-modules then it restricts to a map $V^\alpha \to W^\alpha$ for every weight α. Writing $V = \bigoplus_\alpha V^\alpha$ it means that the short exact sequence is a direct sum of sequences

$$\bigoplus_\alpha (0 \to V_1^\alpha \to V^\alpha \to V_2^\alpha \to 0).$$

Since the whole sequence is exact, each of the summands must be exact.

2. The direct sum decompositions of V and W into weight spaces tensor into the direct sum decomposition shown, and if T acts as scalar multiplication on V^α and W^β via weights α, β, then it acts on $V^\alpha \otimes_F V^\beta$ as scalar multipliction, via the weight $\alpha+\beta$.

3. The idempotents ξ_α $S_F(n,r)$ and $S_L(n,r)$ may be identified, because they have the same definition in both cases. We see that $(L \otimes_F V)^\alpha = \xi_\alpha(L \otimes_F V) = L \otimes_F \xi_\alpha V = L \otimes_F V^\alpha$. □

If V is an $S_F(n,r)$-module its *formal* character is

$$\Phi_V(X_1, \dots, X_n) = \sum_{\alpha \in \Lambda(n,r)} \dim V^\alpha X_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Example 4.1.12. When $V = F$ is the trival module, so $V = V^{(0,0,0,\dots)}$ we have $\Phi_V = 1$.

If V is the natural representation F^n then $\Phi_V = X_1 + \cdots + X_n$.

If $V = \bigwedge^r(E)$ then $\Phi_V = \sum_{J \subseteq \{1,\dots,n\}, |J|=r} (\prod_{j \in J} X_j)$ is the rth elementary symmetric function.

Proposition 4.1.13. *1. Φ_V is symmetric. In fact,*

$$\Phi_V = \sum_{\lambda \in \Lambda^+(n,r)} m_\lambda(X_1, \dots, X_n)$$

where m_λ is the monomial symmetric function, the sum of the orbit of $X_1^{\lambda_1} \cdots X_n^{\lambda_n}$ under W.

2. *If $0 \to V_1 \to V \to V_2 \to 0$ is a short exact sequence of $S_F(n,r)$-modules then $\Phi_V = \Phi_{V_1} + \Phi_{V_2}$.*

3. *If V and W are polynomial representations then $\Phi_{V\otimes W} = \Phi_V \cdot \Phi_W$.*

4. *If $L \supseteq F$ is a field extension then $\Phi_{L\otimes_F V} = \Phi_V$.*

Corollary 4.1.14. *The additive subgroup of $\mathbb{Z}[X_1,\dots,X_n]$ generated by the Φ_V, where V is polynomial of degree r, is the space of all symmetric functions of degree r.*

Proof. It contains all products of elementary symmetric functions of degree r. $\square$

For a polynomial representation of degree r and an element $g \in GL_n(F)$ we now define $\phi_V(g) = \operatorname{Trace}\rho(g) \in F$. Thus $\phi_V = \operatorname{Trace}(\rho_{i,j}) \in A(n,r)$ is a polynomial function that is the trace of the invariant matrix of ρ and because of this we see that it is independent of any choice of basis. The same is not so immediately clear for the formal character Φ_V.

Theorem 4.1.15. *Let V be a polynomial representation of degree r. Then $\phi_V(g) = \Phi_V(\zeta_1,\dots,\zeta_n)$ where $\zeta_1,\dots,\zeta_n$ are the eigenvalues of g in some suitable field extension of F, taken with multiplicities according to their generalized eigenspaces.*

It does not matter in which order we take $\zeta_1,\dots,\zeta_n$ since Φ_V is symmetric.

Proof. Both ϕ_V and Φ_V are unchanged under field extension, and so we may assume the field F is algebraically closed.

If $g \in GL_n(F)$ is a diagonlizable element then there is a basis of the natural module with respect to which g is diagonal. There is now a base change matrix sending this basis to the standard basis and hence an element $z \in GL_n(F)$ so that $zgz^{-1} \in T$. The eigenvalues of g and zgz^{-1} are the same, and so $\phi_V(g) = \phi_V(zgz^{-1}) = \Phi_V(\zeta_1,\dots,\zeta_n)$, since the formula is clearly true for elements of T.

Now the diagonalizable elements contain the elements with distinct eigenvalues, and these form a dense subset of $GL_n(F)$ (namely, the points where the discriminant of the characteristic polynomial of the invariant matrix is nonzero). It follows that $\phi_V(g) = \Phi_V(\zeta_1,\dots,\zeta_n)$ for all $g \in GL_n(F)$. $\square$

What is wrong with the argument given to prove the theorem?

In his notes, Green adopts the following line of argument to prove the last theorem. Let $C = (\rho_{i,j})$ be the invariant matrix of ρ, let u be an indeterminate, and put

$$\det(uI - C) = u^n - f_1u^{n-1} + \cdots \pm f_0$$

in $A(n,r)$, so that $f_s = \pm\phi_{\bigwedge^s V} = \underline{e}_s$ is the degree s elementary symmetric function. We have $f_s(g) = \underline{e}_s(\zeta_1,\dots,\zeta_n)$. We may write $\Phi_V = \sum_\mu b_\mu \underline{e}_1^{\mu_1}\cdots\underline{e}_r^{\mu_r}$ for some $b_\mu \in \mathbb{Z}$. Put $\psi = \sum_\mu b_\mu f_1^{\mu_1}\cdots f_r^{\mu_r} \in A(n,r)$. Now $\Phi_V(\zeta_1,\dots,\zeta_n) = \psi(g)$. If $zgz^{-1} = \operatorname{diag}(\zeta_1,\dots,\zeta_n)$ acts via the weight spaces $V = \bigoplus V^\alpha$ then each V^α contributes $\dim V^\alpha$ terms $\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n}$ and $\phi_V(g) = \phi_V(zgz^{-1}) = \Phi_V(\zeta_1,\dots,\zeta_n)$. We deduce that $\psi = \phi_V$ on all diagonalizable elements, and hence on all $g \in GL_n(F)$, since the diagonalizable elements are dense.

Theorem 4.1.16 (Frobenius and Schur)**.** *Let $V_1, \dots, V_t$ be absolutely simple, non-isomorphic, finite dimensional representations of an algebra A over a field F. Take bases for $V_1, \dots V_t$, giving coordinate functions $f_{ij}^{(k)} : A \to F$, $1 \le k \le t$. Then the functions $f_{ij}^{(k)}$ are linearly independent.*

Note that V_i is absolutely simple if and only if $\mathrm{End}_A(V_i) = F$.

Proof. Consider the homomorphism $A \to \mathrm{End}_F(V_1 \oplus \cdots \oplus V_t)$ given by the representation on the direct sum. Its image is a finite dimensional algebra A_1, and as A_1-modules $V_1, \dots, V_t$ are also absolutely simple and non-isomorphic. It suffices to replace A by A_1 in the argument. Now $\mathrm{Rad}(A_1) \cdot V_i = 0$ for all i, and since A_1 acts faithfully on $V_1 \oplus \cdots \oplus V_t$ we get $\mathrm{Rad}(A_1) = 0$ and A_1 is semisimple. Again because the action is faithful, $V_1, \dots, V_t$ is a complete set of simple modules. Each simple component of A_1 is a matrix algebra $M_{n_i}(\mathrm{End}_{A_1}(V_i))$ where $n_i = \dim_{\mathrm{End}(V_i)} V_i$. It follows that $A_1 = \mathrm{End}_F(V_1 \oplus \cdots \oplus V_t)$. From this we see that the coordinate functions are independent. □

Corollary 4.1.17. *Let $\Phi_1, \dots, \Phi_t$ be the formal characters of a set of mutually non-isomorphic, absolutely simple $S_F(n,r)$-modules $V_1, \dots, V_t$. Then $\Phi_1, \dots, \Phi_t$ are linearly independent elements of the ring of symmetric polynomials.*

Proof. By the Frobenius-Schur Theorem 4.1.16, the natural characters $\phi_1, \dots, \phi_t$ of $V_1, \dots, V_t$ are linearly independent elements of $A(n,r)$. The $\Phi_1, \dots, \Phi_t$ determine the $\phi_1, \dots, \phi_t$ by Theorem 4.1.15 and so they must be independent also. To put this in symbols, suppose that $a_1\Phi_1 + \cdots + a_t\Phi_t = 0$ is a nonzero linear relation with $a_i \in \mathbb{Z}$. In case the characteristic p of F is finite, we may suppose that p does not divide all the a_i. It follows from Theorem 4.1.15 that $(a_1 1_F)\phi_1(g) + \cdots + (a_t 1_F)\phi_t(g) = 0$ for all Class: Why?
$g \in GL_n(F)$, so that $\phi_1, \dots, \phi_t$ are linearly dependent in $A(n,r)$, a contradiction. □

Lemma 4.1.18. *Let $E \supset F$ be a field extension of finite degree and let A be an F-algebra. Let U and V be A-modules. Then*

$$E \otimes_F \mathrm{Hom}_A(U,V) \cong \mathrm{Hom}_{E \otimes_F A}(E \otimes_F U, E \otimes_F V)$$

via an isomorphism $\lambda \otimes_F f \mapsto (\mu \otimes_F u \mapsto \lambda\mu \otimes_F f(u))$.

Proof. See the homework exercises. The exercise about this is Chapter 9 Exercise 16 from page 190 of my book. □

Theorem 4.1.19. *1. Let μ be a partition of r. The character of*

$$\bigwedge^{\mu} E = \wedge^{\mu_1} E \otimes \wedge^{\mu_1} E \otimes \cdots$$

is $\underline{e}_{\mu_1}\underline{e}_{\mu_2}\cdots$ and has leading term $\mathbf{X}^{\mu'} = X_1^{\mu'_1} X_2^{\mu'_2} \cdots$, where μ' is the conjugate partition of μ.

2. *For all $\lambda \in \Lambda^+(n,r)$ there is an absolutely irreducible module $L_F(\lambda)$ whose formal character $\Phi_{\lambda,F}$ has leading term $X_1^{\lambda_1} X_2^{\lambda_2} \cdots$*

3. *The $\Phi_{\lambda,F}$ where $\lambda \in \Lambda^+(n,r)$ are a $\mathbb{Z}$-basis for the symmetric polynomials of degree r.*

4. *Every irreducible $S_F(n,r)$-module is isomorphic to $L_F(\lambda)$ for exactly one $\lambda \in \Lambda^+(n,r)$.*

Corollary 4.1.20. *Let F, K be two infinite fields of the same characteristic. Then $\Phi_{\lambda,F} = \Phi_{\lambda,K}$.*

Corollary 4.1.21. *The decomposition map is surjective.*

Still needed: the simple modules for $S_F(n,r)$ are parametrized by the $\lambda \in \Lambda^+(n,r)$ for every field F (not just infinite fields). The are all absolutely simple. Conclude that the Young modules are all the summands of the M^λ as FS_r-modules, and that the Y^λ are absolutely indecomposable.

Chapter 5

Connections between the Schur algebra and the symmetric group: the Schur functor

We have already been using two functors between left $S_(n, r)$-modules and right FS_r-modules, both called $\natural$. From the equivalence of categories between projective $S_F(n, r)$-modules and direct sums of summands of $E^{\otimes r}$ as an FS_r-module we obtain the following result. It is amazing that we prove it as a consequence of a substantial development of ideas to do with representations of $GL_n(F)$.

Theorem 5.0.1. *Let F be any field.*

1. *The Young modules Y^λ for FS_r are a complete set of isomorphism types of the indecomposable summands of the permutation modules M^μ. The Young modules are absolutely indecomposable.*

2. *The simple modules for the Schur algebra $S_F(n, r)$ are parametrized by the partitions in $\Lambda^+(n, r)$ and are absolutely simple.*

Proof. Over any infinite field K the simple $S_K(r, r)$-modules are the $L(\lambda)$, parametrized by $\lambda \in \Lambda^+(r, r)$. It follows that over F the number of isomorphism types of simple $S_F(r, r)$ modules is at most the number of partitions of r. This is because if V_1, V_2 are non-isomorphic simple $S_F(n, r)$ modules and K is an infinite field, separable as an extension of F, then $K \otimes_F V_1$ and $K \otimes_F V_2$ have no simple component in common, because they are semisimple and by Lemma 4.1.18. On the other hand we know by the fact that the number of non-isomorphic Young modules equals the number of partitions of r that there are at least that number of simple $S_F(r, r)$-modules. It follows that $S_F(r, r)$ has simple modules parametrized by $\Lambda^+(r, r)$ over any field F, and further more that they are absolutely simple, because on extending by any separable field extension, if any simple is no longer simple then more simple composition factors are introduced.

We deduce at the same time that, over any field F, the M^μ have no more isomorphism types of indecomposable summands than the Y^λ since these isomorphism types biject with the simple $S_F(r,r)$-modules. It also follows that the Y^λ are absolutely indecomposable, or equivalently that $\mathrm{End}_F(Y^\lambda)/\operatorname{Rad}\mathrm{End}_F(Y^\lambda) \cong F$, since this quotient is isomorphic to the endomorphism ring of the corresponding simple $S_F(r,r)$ module. We deduce, finally, that the simple $S_F(n,r)$-modules biject with $\Lambda^+(n,r)$, since this set parametrizes the isomorphism types of indecomposable summands of $E^{\otimes r}$ always. $\square$

There are various functors from $S_R(n,r)$-modules to RS_r-modules. The basic relationship between $S_R(n,r)$ and RS_r is that there is an idempotent $\xi \in S_R(n,r)$ so that $RS_r \cong \xi S_R(n,r)\xi$. We first consider such a situation in abstract.

Proposition 5.0.2. *Let B be a ring containing and idempotent e.*

1. *If V is a left B-module then $eV \cong \mathrm{Hom}_B(Be,V) \cong eB \otimes_B V$ as left eBe-modules.*

2. *$eBe \cong \mathrm{End}_B(Be)^{\mathrm{op}}$ as rings.*

Proof. (1) We have $eV \cong \mathrm{Hom}_B(Be,V)$ via inverse maps $v \mapsto \phi_v$, where $\phi_v(be) = bev$, and $\theta(e) \leftarrow \theta$. The first is a map of left eBe-modules since $exev \mapsto \phi_{exev}$, and $\phi_{exev}(be) = beexev$ and $(exe\phi_v)(be) = \phi_v(beexe) = beexev$. Thus $\phi_{exev} = (exe)\phi_v$.

We also have an isomorphism $eV \cong eB \otimes_B V$ given by inverse maps $ev \mapsto e \otimes_B ev$ and $ebv \leftarrow eb \otimes_B v$, which are again maps of eBe-modules.

(2) Taking $V = Be$ in part (1) we get $eV = eBe = \mathrm{Hom}_B(Be,Be)$ as left eBe-modules, and under this isomorphism, $b \mapsto \phi_b$ where $\phi_b : Be \to Be$ is $\phi_b(xe) = exb$. Now $\phi_b\phi_c(xe) = \phi_b(xec) = xecb = \phi_{cb}(xe)$ and so $\phi_b\phi_c = \phi_{cb}$. Thus we have a ring isomorphism of eBe with $\mathrm{End}_B(Be)^{\mathrm{op}}$. $\square$

Part (2) of the last result says that, in the generality considered there, the three functors $V \mapsto eV$, $\mathrm{Hom}_B(Be,-)$ and $eB \otimes_B -$ are all isomorphic.

In the case of the Schur algebra and the group ring of the symmetric group the idempotent ξ arises as projection onto a summand of $E^{\otimes r}$. The general setup of this kind is that we have a summand Y of a module $\mathcal{E}$ and let

$$e = ip : \mathcal{E} \xrightarrow{p} Y \xrightarrow{i} \mathcal{E}$$

denote the endomorphism of $\mathcal{E}$ that is projection onto Y. Recall that we take the convention that morphisms are applied from the left.

Proposition 5.0.3. *Let A be a ring, let $\mathcal{E}$ be a right A-module and put $B = \mathrm{End}_A(\mathcal{E})$, so that $\mathcal{E}$ is a left B-module. Let Y be a summand of $\mathcal{E}$ as an A-module and let $e \in B$ be the idempotent that is projection onto Y. Then*

1. *$Y \cong e\mathcal{E}$ as right A-modules.*

2. *$eBe \cong \mathrm{End}_A(Y)$ as rings.*

Proof. (1) This is immediate from the definition.

(2) The isomorphism is $\phi \mapsto i\phi p$ with inverse $p\theta i \leftarrow \theta$. □

We now apply this result in the context of the Schur algebra. The isomorphisms work more generally than over a field, so we let R be a commutative ring with 1. By an $S_R(n,r)$-lattice or an RS_r-lattice we mean a module for this ring that is finitely generated and projective as an R-module. Suppose that $n \geq r$. We take for e the idempotent $\xi = \xi_{(1,\ldots,1,0,\ldots,0)}$ in $S_F(n,r)$ that projects $E^{\otimes r}$ onto the weight space $\xi E^{\otimes r} = (E^{\otimes r})^{[1^r]} = M^{[1^r]}$. The *Schur functor* may be viewed as the functor $S_F(n,r)$-mod $\to$ $\xi S_F(n,r)\xi$-mod given by $U \mapsto \xi U$. We will also see that $\xi S_F(n,r)\xi \cong RS_r$ so that the Schur functor becomes a functor $S_F(n,r)$-mod $\to RS_r$-mod. These two module categories are both categories of left modules, and the Schur functor is covariant. We will see that it can also be expressed in terms of the contravariant natural functor $\natural$ introduced previously.

Change F to R in what follows.

Proposition 5.0.4. *Suppose that $n \geq r$ and let $\xi = \xi_{(1,\ldots,1,0,\ldots,0)}$ be the idempotent in $S_F(n,r)$ that projects $E^{\otimes r}$ onto the weight space $\xi E^{\otimes r} = (E^{\otimes r})^{[1^r]} = M^{[1^r]}$. We have isomorphisms as follows.*

1. $\xi S_F(n,r)\xi \cong \mathrm{End}_{RS_r}(\xi E^{\otimes r}) \cong \mathrm{End}_{S_F(n,r)}(S_F(n,r)\xi)^{\mathrm{op}}$ *as rings.*
2. $\xi E^{\otimes r} \cong RS_r$ *as right RS_r-modules.*
3. $\mathrm{End}_{RS_r}(\xi E^{\otimes r}) \cong RS_r$ *as rings.*
4. $\xi S_F(n,r)\xi \cong RS_r$ *as rings.*
5. $E^{\otimes r} \cong S_F(n,r)\xi$ *as $(S_F(n,r), RS_r)$-bimodules.*

Proof. (1) These isomorphisms are part (1) of Proposition 5.0.2 and Proposition 5.0.3, translated to the present context.

(2) The weight space $\xi E^{\otimes r}$ is generated as an RS_r module by the basic tensor $e_1 \otimes e_2 \otimes \cdots \otimes e_r$ and so has a permutation basis permuted regularly by S_r, from which the isomorphism follows.

(3) is immediate from (2) using a standard identification of the endomorphism ring of the regular representation. Since we follow the convention that morphisms are applied from the left, we do not take an opposite ring in this isomorphism. A specific isomorphism may be obtained by letting an element $y \in RS_r$ correspond to the endomorphism ϕ_y of $\xi E^{\otimes r}$ specified by $\phi_y(e_1 \otimes \cdots \otimes e_r \cdot z) = e_1 \otimes \cdots \otimes e_r \cdot yz$.

(4) comes by combining (3) and (1).

(5) In the first place $S_F(n,r)\xi$ is a right $\xi S_F(n,r)\xi$-module, but we interpret it as a right RS_r-module using the isomorphism of (4). Taking $S_F(n,r)$ as $\mathrm{End}_{RS_r}(E^{\otimes r})$ the subset $S_F(n,r)\xi$ is isomorphic to $\mathrm{Hom}_{S_r}(\xi E^{\otimes r}, E^{\otimes r})$ via an isomorphism that associates $\theta\xi \mapsto \theta i$ where $\xi = i \circ p$ is the factorization as projection followed by inclusion of the summand $\xi E^{\otimes r}$. By (2) this is isomorphic to $\mathrm{Hom}_{RS_r}(RS_r, E^{\otimes r}) \cong E^{\otimes r}$. We check that these isomorphisms preserve the bimodule actions.

□

We have been interested in the Schur algebra here, but note that if we take a permutation summand M^λ of $E^{\otimes r}$ other than $\xi E^{\otimes r} \cong M^{[1^r]}$, its endomorphism ring over RS_r would be the Hecke algebra $\mathrm{End}_{RS_r}(R\uparrow_{S_\lambda}^{S_r})$. A similar analysis would apply in this situation.

The reader may wonder why a particular functor was chosen as the Schur functor when there are other functors available that might merit equal consideration. We conclude this section with some technical identities that show that the other functors would have done just as well.

Corollary 5.0.5. *As above, let $\xi = \xi_{[1^r]}$ and let V be a left $S_F(n,r)$-module. The Schur functor $V \mapsto \xi V$ may be described in the following ways:*

$$\xi V \cong \xi S_R(n,r) \otimes_{S_R(n,r)} V \cong \mathrm{Hom}_{S_F(n,r)}(S_F(n,r)\xi, V) \cong \mathrm{Hom}_{S_F(n,r)}(E^{\otimes r}, V)$$

as left RS_r-modules.

Proof. This follows from Proposition 5.0.2 and part (5) of Proposition 5.0.4. □

Previously we have used the functors $X \to X^\natural = \mathrm{Hom}(-, E^{\otimes r})$ in both directions between right FS_r-modules and left $S_F(n,r)$-modules, with homomorphisms being taken over the appropriate ring. There are several other functors between left $S_F(n,r)$-modules and right FS_r-modules. We describe some of these now and show that they are all expressible in terms of each other. If X is a right RS_r-lattice we will let $\hat{X}$ denote the left RS_r-lattice with the same set as X, and with the left action of S_r given by $\pi \cdot x = x\pi^{-1}$. We similarly use ˆ to change a left RS_r-lattice into a right lattice, and also to change the side of $S_R(n,r)$ lattices using the antiautomorphism of $S_R(n,r)$.

Proposition 5.0.6. *Let R be a commutative ring with 1. Let X be a right RS_r-lattice and V a left $S_R(n,r)$-lattice. Then*

1. $X^\natural = \mathrm{Hom}_{RS_r}(X, E^{\otimes r}) \cong \widehat{\mathrm{Hom}_{RS_r}(E^{\otimes r}, X^*)} \cong (E^{\otimes r} \otimes_{RS_r} \hat{X})^\circ$ *as left $S_R(n,r)$-modules.*
2. $V^\natural = \mathrm{Hom}_{S_F(n,r)}(V, E^{\otimes r}) \cong \widehat{\mathrm{Hom}_{S_F(n,r)}(E^{\otimes r}, V^\circ)} \cong \widehat{\xi S_R(n,r) \otimes_{S_R(n,r)}} V^\circ$ *as right RS_r-modules*

In cases where R is a principal ideal domain for instance, observe that $(E^{\otimes r} \otimes_{RS_r} \hat{X})^\circ$ will be torsion free, even though $E^{\otimes r} \otimes_{RS_r} \hat{X}$ might not be.

Proof. We use the fact that $E^{\otimes r}$ is self dual as a $(S_R(n,r), RS_r)$-bimodule. Thus $\mathrm{Hom}_{RS_r}(X, E^{\otimes r}) \cong \mathrm{Hom}_{RS_r}(E^{\otimes r*}, X^*) \cong \mathrm{Hom}_{RS_r}(E^{\otimes r}, X^*)$ as R-modules. The first is a left $S_R(n,r)$-module and the last is a right $S_R(n,r)$-module. We check that by changing the module side we obtain a module isomorphism, and this demonstrates the first isomorphism in (1). A similar approach proves the first isomorphism in (2). The final isomorphism of (2) was seen already in Corollary 5.0.5.

Check these things: they might not be right!

To prove the final part of (1) we start with

$$\mathrm{Hom}_R(E^{\otimes r} \otimes_{RS_r} \hat{X}, R) \cong \mathrm{Hom}_{RS_r}(E^{\otimes r}, \mathrm{Hom}_R(\hat{X}, R)) = \mathrm{Hom}_{RS_r}(E^{\otimes r}, X^*)$$

by adjointness. Naturality implies that this an isomorphism of right $S_R(n,r)$-modules. Applying $\hat{}$ to both sides yields the result. □

5.1 The general theory of the functor $f : B$-mod $\to eBe$-mod

The systematic exposition of this functor seems to have first appeared in Green's notes. We let B be an R-algebra and $e^2 = e \in B$ an idempotent. We denote by $f : B$-mod $\to$ eBe-mod the operation $f(V) = eV$.

This needs to be presented with the preceding results.

Proposition 5.1.1. *1. $eV \cong \mathrm{Hom}_B(Be, V) \cong eB \otimes_B V$.*

2. f is a functor. It has left adjoint $U \mapsto Be \otimes_{eBe} U$ and right adjoint $\mathrm{Hom}_{eBe}(eB, V)$.

3. f is exact.

4. If $V \in B$-mod is simple then eV is either zero or simple in eBe-mod.

We construct a functor $h : eBe$-mod $\to B$-mod given by $W \mapsto Be \otimes_{eBe} W$ with the action of B given by left multiplication and if $\psi : W \to W'$ then $h(\psi) = 1_{Be} \otimes \psi$.

Class: are the left and right adjoints of f naturally isomorphic? (No)

Proposition 5.1.2. *Let $W \in eBe$-mod. Then $fh(W) \cong W$.*

Example 5.1.3. For $S_{\mathbb{F}_2}(2,2)$ we have $h(\mathbb{F}_2) = \begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}$.

More needs to be filled in here, following Green's notes.

Example 5.1.4. Letting $h(S^{[r]}) = E^{\otimes r} \otimes_{FS_r} S^{[r]} = \mathrm{Sym}^r(E)$. We deduce that $\mathrm{Sym}^r(E)$ has a unique simple quotient as an $S_F(n,r)$-module.

5.2 Applications

Let $N \geq n$ and let E_N be a vector space with basis $e_1, \ldots, e_N$. Let E_n be the subspace spanned by $e_1, \ldots, e_n$.

Lemma 5.2.1. *$E_n^{\otimes r}$ is a direct summand of $E_N^{\otimes r}$ as a FS_r-module.*

Proof. $E_N^{\otimes f}$ is the span of basic tensors $e_{i_1} \otimes \cdots \otimes e_{i_r}$ with $i_j \leq n$ for all j, together with the remaining basic tensors that do not satisfy this condition. These two sets are preserved under the action of S_r. □

Let $e : E_N^{\otimes r} \to E_n^{\otimes r}$ be projection with respect to the direct sum decomposition indicated in the last lemma.

Lemma 5.2.2. $eS_F(N,r)e \cong S_F(n,r)$

Proof. This is immediate from Proposition 5.0.3. $\square$

Proposition 5.2.3. *Let $N \geq n \geq r$. the functors $f : S_F(N,r)\text{-mod} \to S_F(n,r)\text{-mod}$ and $h : S_F(n,r)\text{-mod} \to S_F(N,r)\text{-mod}$ are inverse, up to natural equivalence. The numbers of isomorphism types of simple modules in the two categories are equal.*

Proof. The numbers of simple modules in the two categories are equal and $fh(W) \cong W$ always. From this it follows that f induces a bijection on the simple modules. In particular, if V is a (nonzero) simple module then $f(V) \neq 0$. Furthermore f is exact, so if W is simple and $h(W)$ is not simple, then neither is $fh(W)$, a contradiction. It follows that $h(W)$ must be simple. We see that $h^* = h$, because $1-e$ kills no simple module V. For any module V the natural map $hf(V) = Be \otimes_{eBe} eV \to V$ is an isomorphism because $fhf(V) = f(V)$ has the same length as V and $hf(V) \to V$ induces an isomorphism on applying f. $\square$

Refer to results not yet written.

Two rings whose module categories are equivalent are said to be *Morita equivalent*. The Morita equivalence of $S_F(N,r)$ and $S_F(n,r)$ when $N \geq n \geq r$ just proven may also be seen from the general theory of Morita equivalence, on observing that these two rings are the endomorphism rings of modules with the same indecomposable summands, taken with different multiplicities.

Theorem 5.2.4. *Let E be a vector space of dimension n over the field F. Provided $n \geq r$, the lattice of $S_F(n,r)$-submodules of each of $E^{\otimes r}$, $ST^r(E)$, $\mathrm{Sym}^r(E)$, $\Lambda^r(E)$ and $\Lambda^{\lambda_1}E \otimes \cdots \otimes \Lambda^{\lambda_d}E$ is independent of n. In each case, the functor $V \mapsto eV$ gives an isomorphism of the lattices. In particular, $\Lambda^r(E)$ is a simple $S_F(n,r)$-module for all $n \geq r$.*

Proof. It is immediate that the operation $e : E_N^{\otimes r} \to E_n^{\otimes r}$ sends each of the subspaces of $E_N^{\otimes r}$ mentioned in the list to the corresponding subspace of $E_n^{\otimes r}$. This is because e sends every basic tensor $e_1 \otimes \cdots \otimes e_N$ to zero unless all e_i lie in E_n, so that symmetric tensors are sent to symmetric tensors, and so on. We realize $\Lambda^r(E_N$ as the span of elements $\sum_{\pi \in S_r} (-1)^{\mathrm{sign}(\pi)} e_{i_{1\pi}} \otimes \cdots \otimes e_{i_{r\pi}}$, and these elements for E_N are sent to the similar elements for E_n by e. From this we obtain the isomorphism of lattices, by Proposition 5.2.3. Since $\Lambda^r(E_r)$ has dimension 1 it is simple, and hence so are all the $\Lambda^r(E)$. $\square$

We conclude with the observation that, in case the field is infinite, the weight description of simple modules is compatible with the functorial correspondence between representations of different general linear groups. When $N \geq n \geq r$ the set of weights $\Lambda^+(n,r) = \Lambda^+(N,r)$ is the same. If λ is such a weight we have an idempotent ξ_λ in the corresponding Schur algebra. Let us write $\xi_{\lambda,S(n,r)}$ for this idempotent in the algebra $S(n,r)$. As above, we have $S_F(n,r) \cong eS_F(N,r)e \subseteq S_F(N,r)$ and by this means we may regard $S_F(n,r)$ as a subset of $S_F(N,r)$. We will also write $L_{S(N,r)}(\lambda)$ for the corresponding simple $S_F(N,r)$-module.

Proposition 5.2.5. *Let F be an infinite field. Let $N \geq n \geq r$ and let $e : E_N^{\otimes r} \to E_n^{\otimes r}$ be projection as above. Let M be an $S_F(N,r)$-module. We have*

1. $\xi_{\lambda,S_F(N,r)} = e\xi_{\lambda,S_F(N,r)}e = \xi_{\lambda,S_F(n,r)}$.
2. $\dim(eM)^\lambda = \dim M^\lambda$.
3. $eL_{S(N,r)}(\lambda) = L_{S(n,r)}(\lambda)$

Proof. To be supplied. □

Sketch of further applications, to be written fully. We may develop the theory as above without the requirement that $n \geq r$. In this case e kills the simples parametrized by partitions with more than n parts.

We also need to develop the theory of the original Schur functor $S_F(n,r)$-mod $\to$ FS_r-mod. In characteristic 0 or $p > r$ the two categories have the same number of isomorphism classes of simples and so f and h are inverse equivalences. In general, there is a decomposition map that expresses simple modules in characteristic 0 as a sum of simple modules in characteristic p, in a Grothendieck group. This can also be described by expression the formal character of a simple module in characteristic 0 as a sum of formal characters of simple modules in characteristic p. The Schur functor is defined the same way in all characteristics and over the integers, so it commutes with the decomposition map. Since it is exact, it implies that the decomposition matrix of the symmetric group is a submatrix of the decomposition matrix of the Schur algebra.

As part of this we need to reconcile the labelling of simple modules for $S_F(n,r)$ and FS_r across the correspondence given by the Schur functor.

Example 5.2.6. In characteristic 0 we have formal characters $\Phi_{[2],0} = X^2 + XY + Y^2$ and $\Phi_{[1^2],0} = XY$. In characteristic 2 we have $\Phi_{[2],2} = X^2 + Y^2$ and $\Phi_{[1^2],2} = XY$. Thus $\Phi_{[2],0} = \Phi_{[2],2} + \Phi_{[1^2],2}$. This is the formal character of the symmetric square $\mathrm{Sym}^2(E_2)$ which is simple in characteristic 0, but has two composition factors with the characters shown in characteristic 2, as seen in earlier calculations. Also $\Phi_{[1^2],0} = \Phi_{[1^2],2}$. This is the formal character of $\Lambda^2(E_2)$, which is simple in all characteristics. More calculations of characters could be done as exercises.

$S_{\mathbb{F}_2}(2,2)\xi = \begin{smallmatrix}\beta\\ \alpha\\ \beta\end{smallmatrix} = \begin{smallmatrix}L[1^2]\\ L[2]\\ L[1^2]\end{smallmatrix}$ and $\Lambda^2(E) = \beta = L[1^2]$. $Y^{[2]} = k, Y^{[2]\natural} = \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}$ and $Y^{[1^2]} = \begin{smallmatrix}k\\ k\end{smallmatrix}$, $Y^{[1^2]\natural} = \begin{smallmatrix}\beta\\ \alpha\\ \beta\end{smallmatrix}$.

Chapter 6

Representations of the category of vector spaces

If $\mathcal{C}$ is a category and R is a commutative ring we let $\mathrm{Rep}(\mathcal{C}, R)$ be the category of functors $\mathcal{C} \to R$-mod.

Example 6.0.1. Let G be a group and $\mathcal{F}$ the corresponding category, so $\mathcal{F}$ has a single object $*$ and $\mathrm{Hom}(*, *) = G$, with composition of morphisms equal to multiplication within the group. Then $\mathrm{Rep}(\mathcal{F}, R)$ identifies with the category of representations of G, and with RG-mod.

Example 6.0.2. Let F be a field and let $\mathcal{C} = F$-mod be the category of finite dimensional vector spaces over F. We study $\mathrm{Rep}(F\text{-mod}, F)$, the category of functors from finite dimensional vector spaces over F to finite dimensional vector spaces over F. For an F-vector space E, let us write $T^r(E) = E^{\otimes r}$. Then T^r, ST^r, Sym^r, Λ^r are all examples of functors in $\mathrm{Rep}(F\text{-mod}, F)$.

A morphism of functors is a natural transformation. For any category $\mathcal{C}$, if M and N are functors $\mathcal{C} \to R$-mod we say that M is a *subfunctor* of N if $M(x) \subseteq N(x)$ for all objects x of $\mathcal{C}$ in such a way that M is a functor in its own right when R-module homomorphisms are restricted from $N(x)$ to $M(x)$. In other words, the inclusion maps give a natural transformation $M \to N$.

Example 6.0.3. Thus ST^r is a subfunctor of T^r. This requires us to check that every linear map $\theta : E \to E'$ induces a map $E^{\otimes r} \to E'^{\otimes r}$ that sends $ST^r(E) \to ST^r(E')$.

If M is a subfunctor of N we may define a *quotient functor* N/M by $(N/M)(x) = N(x)/M(x)$, with morphisms induced on these quotient modules from the morphisms between the values of N. A sequence of morphisms of functors is *exact* if on each evaluation at objects x of $\mathcal{C}$ the sequence of modules is exact. We say that a functor in $\mathrm{Rep}(\mathcal{C}, R)$ is *simple* if it has no subfunctors other than the zero functor or itself.

Example 6.0.4. When F is a field of characteristic 2 there are natural short exact sequences $0 \to \Lambda^2(E) \to ST^2(E) \to L([2])(E) \to 0$. This means there is a short exact sequence of functors $0 \to \Lambda^2 \to ST^2 \to L([2]) \to 0$. We will see that both Λ^2 and $L([2])$ are simple functors.

Given an object $x \in \mathcal{C}$ and a representation $M \in \mathrm{Rep}(\mathcal{C}, R)$ we find that, for each object x, the evaluation $M(x)$ is a representation of the monoid $\mathrm{End}_{\mathcal{C}}(x)$, meaning a homomorphism of monoids $\mathrm{End}_{\mathcal{C}}(x) \to \mathrm{End}_R(M(x))$. This is the same thing as a module for the *monoid algebra* $R\,\mathrm{End}_{\mathcal{C}}(x)$. Representations of categories have a number of properties in common with representations of groups. There is *constant functor* at R, namely the functor $\mathcal{C} \to R$-mod that takes the value R on all objects, and in which every morphism acts as the identity morphism. When R is a field, it is immediate that the constant functor at R is simple.

There is also an internal *tensor product* of representations. If M and N are both representations of $\mathcal{C}$ we define $M \otimes_R N$ to be the representation given by $(M \otimes_R N)(x) := M(x) \otimes_R N(x)$ on objects, and if $\alpha : x \to y$ is a morphism in $\mathcal{C}$ then $(M \otimes_R N)(\alpha) := M(\alpha) \otimes_R N(\alpha)$.

6.1 Simple representations of the matrix monoid

We examine the particular case when $M \in \mathrm{Rep}(F\text{-mod}, F)$ is a functor from F-vector spaces to F-vector spaces. When E is a vector space of dimension n over F, then $M(E)$ is a representation of $\mathrm{Mat}_n(F)$, considered as a monoid under multiplication. By further restriction we see that $M(E)$ is also a representation of $GL_n(F)$.

Example 6.1.1. Consider the functor in $\mathrm{Rep}(F\text{-mod}, F)$ that assigns the value F to each vector space E, including the zero vector space, and the identity map to each morphism of vector spaces. The endomorphism monoid of the zero vector space consists of a single element 0, but it does not act as zero on the value of the functor at the zero vector space.

Because of this connection between representations of F-mod and representations of $\mathrm{Mat}_n(F)$, we start be considering these latter representations. To this end, let

$$e = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \in \mathrm{Mat}_n(F).$$

Example 6.1.2. We describe two ways in which, for a vector space E, a space such as $E^{\otimes r}$ can be regarded as a representation of $\mathrm{Mat}_n(F)$. The first is the usual way, in which every $n \times n$ matrix actors via its r-fold tensor power. The second is the same as the first for non-singular matrices. We let all singular matrices act as zero. The fact that this gives a representation of $\mathrm{Mat}_n(F)$ relies on the fact that the composite of a singular matrix with any matrix is still a singular matrix. In this second representation, the idempotent e acts as 0.

Lemma 6.1.3. *1.* $\mathrm{Mat}_n\, e\, \mathrm{Mat}_n$ *is the set of matrices of rank less than* n*.*

2. *$F\,\mathrm{Mat}_n\, eF\,\mathrm{Mat}_n$ is an ideal in $F\,\mathrm{Mat}_n$ with quotient isomorphic to $FGL_n(F)$.*

3. *$e\,\mathrm{Mat}_n\, e \cong \mathrm{Mat}_{n-1}$.*

Proof. To be supplied. □

Proposition 6.1.4. *Let V be a simple $F\,\mathrm{Mat}_n$-module. Then either*

1. *V is a simple $FGL_n(F)$-module on which Mat_{n-1} acts as 0, or*

2. *V restricts to give a simple $F\,\mathrm{Mat}_{n-1}$-module.*

Proof. To be supplied. □

Corollary 6.1.5. *The simple $F\,\mathrm{Mat}_n$-modules biject with the union of the simple $FGL_j(F)$-modules where $0 \le j \le n$.*

According to Nick Kuhn, when $j < n$ each simple representation of Mat_n corresponding to a simple $FGL_j(F)$-module is simple when restricted to $GL_n(F)$. Write an account of this here.

Example 6.1.6. Class exercises: How many elements do $\mathbb{F}_2\,\mathrm{Mat}_0(\mathbb{F}_2)$, $\mathbb{F}_2\,\mathrm{Mat}_1(\mathbb{F}_2)$ and $\mathbb{F}_2\,\mathrm{Mat}_2(\mathbb{F}_2)$ have? How many simple representations do they each have? Are any of them semisimple? (That might need to come after the next example.)

Example 6.1.7. Write $I_j \in \mathrm{Mat}_n(F)$ for the matrix that has j entries 1 down the leading diagonal and 0 elsewhere. Mat_0 has only a single element: the 0×0 matrix. Thus $F\,\mathrm{Mat}_0(F) \cong F$ as rings and its modules are vector spaces. Mat_1 has two elements which we may write as I_0 and I_1. Thus $F\,\mathrm{Mat}_1(F) = FI_0 \oplus F(I_1 - I_0)$ as rings, and it is the direct sum of two fields, hence semisimple. The first summand gives a representation $S_{0,F}$, and since $I_0 I_0 = I_1 I_0 = I_0$ both I_0 and I_1 act as 1 on this representation. This representation has the form $\mathrm{Mat}_1(F)I_0 \otimes_{F\,\mathrm{Mat}_0(F)} S_{0,F}$, as we may check. The second summand gives a representations $S_{1,F}$ on which I_1 acts as 1 and I_0 acts as 0. Note that $GL_1(F)$ acts as 1 on both these simples.

Next $\mathrm{Mat}_2(\mathbb{F}_2)$ has 16 elements, of which 6 are invertible, 9 have rank 1 and 1 has rank 0. We calculate $F\,\mathrm{Mat}_2\, I_1 \otimes_{F\,\mathrm{Mat}_1(F)} S_{1,F}$. For this, $F\,\mathrm{Mat}_2\, I_1$ has basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts by sending

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We calculate that $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \otimes_{F\,\mathrm{Mat}_1(F)} S_{1,F} = 0$. Thus $F\,\mathrm{Mat}_2\, I_1 \otimes_{F\,\mathrm{Mat}_1(F)} S_{1,F}$ is a 2-dimensional space spanned by $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \otimes_{F\,\mathrm{Mat}_1(F)} x$ and $\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \otimes_{F\,\mathrm{Mat}_1(F)} x$, where x spans $S_{1,F}$. On this space $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ acts as usual. We see that

$$F\,\mathrm{Mat}_2\, I_1 \otimes_{F\,\mathrm{Mat}_1(F)} S_{1,F} = S_{1,F}$$

regarded now as a representation of $\mathrm{Mat}_2(F)$. By a similar calculation we see that $F\,\mathrm{Mat}_2\, I_1 \otimes_{F\,\mathrm{Mat}_1(F)} S_{0,F}$ has dimension 1, and has all matrices acting as the identity (including the zero matrix). Thus this representation is $S_{0,F}$ as a representation of $\mathrm{Mat}_2(F)$. These simple representations turn out to be induced from smaller matrix monoids.

Are all simple representations induced from their minimal space?

6.2 Simple representations of categories and the category algebra

As mentioned in the introduction, when $\mathcal{C}$ is a small category and R is a commutative ring with a 1 we define a *representation* of $\mathcal{C}$ over R to be a functor $F : \mathcal{C} \to R$-mod where R-mod is the category of R-modules. Such a representation may be identified as a module for a certain algebra which we now introduce. We define the *category algebra* $R\mathcal{C}$ to be the free R-module with the morphisms of the category $\mathcal{C}$ as a basis. The product of morphisms α and β as elements of $R\mathcal{C}$ is defined to be

$$\alpha\beta = \begin{cases} \alpha \circ \beta & \text{if } \alpha \text{ and } \beta \text{ can be composed} \\ 0 & \text{otherwise} \end{cases}$$

and this product is extended to the whole of $R\mathcal{C}$ using bilinearity of multiplication. We have constructed an associative algebra which can be found in Section 7 of [4] (where the approach is to pass through an intermediate step in which we first 'linearize' $\mathcal{C}$). Our convention is that we compose morphisms on the left, so that if the domain $\mathrm{dom}(\alpha)$ equals the codomain $\mathrm{cod}(\beta)$ then we obtain a composite $\alpha \circ \beta$. Because of this we will work almost entirely with left modules when we come to consider modules for the category algebra.

If $\mathcal{C}$ happens to be a group, that is a category with one object in which every morphism is invertible, then a representation of $\mathcal{C}$ is the same thing as a representation of the group in the usual sense, namely a group homomorphism from the group to the group of automorphisms of an R-module, and the category algebra $R\mathcal{C}$ is the *group algebra*. It is a familiar fact that group representations may be regarded as the same thing as modules for the group algebra, and we will see that something similar holds with categories in general. One of the themes of this account is that representations of categories share a number of the properties of group representations.

When the category happens to be a partially ordered set the category algebra $R\mathcal{C}$ is known as the *incidence algebra* of the poset. Indeed, this may be taken as a definition of the incidence algebra.

The third example is that of representations of a quiver. A *quiver* Q is a directed graph, and given such data we may form the free category FQ on Q, which is the category whose objects are the vertices of Q and whose morphisms are all the possible composites of the arrows in Q (including for each object a composite of length zero which is the identity morphism at that object). The category algebra RFQ is the same as the path algebra of Q, and it is well known that, provided Q has finitely many vertices, modules for the path algebra may be identified with representations of the quiver.

Our first result says that representations of $\mathcal{C}$ are the same thing as $R\mathcal{C}$-modules in general, at least when $\mathcal{C}$ has finitely many objects.

Proposition 6.2.1 ([4])**.** *Let $\mathcal{C}$ be a small category, let $(R\text{-mod})^{\mathcal{C}}$ be the category of representations of $\mathcal{C}$ and let $R\mathcal{C}$-mod be the category of $R\mathcal{C}$-modules. There are functors $r : (R\text{-mod})^{\mathcal{C}} \to R\mathcal{C}\text{-mod}$ and $s : R\mathcal{C}\text{-mod} \to (R\text{-mod})^{\mathcal{C}}$ with the properties that*

1. *$sr \cong 1_{(R\text{-mod})^{\mathcal{C}}}$, and*
2. *r embeds $(R\text{-mod})^{\mathcal{C}}$ as a full subcategory of $R\mathcal{C}$-mod, and if $\mathcal{C}$ has finitely many objects then $rs \cong 1_{R\mathcal{C}\text{-mod}}$.*

Thus if $\mathcal{C}$ has finitely many objects the representations of $\mathcal{C}$ over R may be identified with $R\mathcal{C}$-modules.

Proof. The idea is the same as the identification of group representations with modules for the group algebra, with an extra ingredient. Given a representation $M : \mathcal{C} \to R$-mod we obtain an $R\mathcal{C}$-module $r(M) = \bigoplus_{x \in \mathrm{Ob}\mathcal{C}} M(x)$ where the action of a morphism $\alpha : y \to z$ on an element $u \in M(x)$ is to send it to $M(\alpha)(u)$ if $x = y$ and zero otherwise (applying morphisms from the left.) Conversely, given an $R\mathcal{C}$-module U, for each $x \in \mathrm{Ob}\mathcal{C}$ let 1_x denote the identity morphism at x and define a functor $M = s(U) : \mathcal{C} \to R$-mod by $M(x) = 1_x U$. If $\alpha : x \to z$ is a morphism in $\mathcal{C}$ and $u \in 1_x U$ we define $M(\alpha)(u) = \alpha u$. The two functors r and s evidently have the properties claimed, and in case $\mathcal{C}$ has finitely many objects they give an equivalence of categories between representations of $\mathcal{C}$ over R and $R\mathcal{C}$-modules. ☐

Example 6.2.2. Let $\mathcal{C}$ be the category with two objects, x and y, and with morphisms 1_x, 1_y, $\alpha : x \to y$, $\beta : y \to x$ and $\gamma : y \to y$, satisfying $\beta\alpha = 1_x$ and $\alpha\beta = \gamma$. From this it follows that $\gamma^2 = \gamma$, $\beta\gamma = \beta$ and $\gamma\alpha = \alpha$.

$$\begin{tikzcd}
x \arrow[loop left, "1_x"] \arrow[r, bend left, "\alpha"] & y \arrow[l, bend left, "\beta"] \arrow[loop above, "1_y"] \arrow[loop below, "\gamma"]
\end{tikzcd}$$

This category will be considered again in Example 6.3.3. Since $\mathcal{C}$ has 5 morphisms, the rank of the category algebra $R\mathcal{C}$ is 5. Consider the constant functor $M : \mathcal{C} \to R$-mod that takes the value R at each object x and y, and sends each morphism in $\mathcal{C}$ to the identity morphism. This functor corresponds to an $R\mathcal{C}$-module that is free as an R-module of rank 2, and where the morphisms 1_x, 1_y, α, β, γ act via the matrices

$$\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\1&0\end{bmatrix},\quad \begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

Assume that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ and that $\mathcal{D}$ has finitely many objects. In this situation the category algebra of $\mathcal{D}$ is a subalgebra of the category algebra of $\mathcal{C}$, and in fact $R\mathcal{D} = 1_{R\mathcal{D}}R\mathcal{C}1_{R\mathcal{D}}$. Restriction from $\mathcal{C}$ to $\mathcal{D}$ is a functor which sends an $R\mathcal{C}$-module M to $M\downarrow^{\mathcal{C}}_{\mathcal{D}}= 1_{R\mathcal{D}}M$. Its left adjoint is $N \mapsto N\uparrow^{\mathcal{C}}_{\mathcal{D}}= R\mathcal{C}\otimes_{R\mathcal{D}} N$, and the right adjoint of restriction is $N \mapsto \mathrm{Hom}_{R\mathcal{D}}(R\mathcal{C}, N)$.

We studied the relationship between algebras B and eBe where e is an idempotent in B in Section 5.1. The properties of this relationship have an immediate consequence for representations of categories.

Proposition 6.2.3 (see [2])**.** *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ and suppose that $\mathcal{D}$ has finitely many objects. Let M be a representation of $\mathcal{D}$.*

1. *Induction $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ sends projective objects to projective objects. If $\mathcal{E}$ is a set of objects of $\mathcal{D}$ and M is generated by its values on $\mathcal{E}$ then $M\uparrow^{\mathcal{C}}_{\mathcal{D}}$ is also generated by its values on $\mathcal{E}$. Furthermore $M\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}}= M$.*

2. *Restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is an exact functor which sends each simple $R\mathcal{C}$-module either to a simple $R\mathcal{D}$-module or to zero. Every simple $R\mathcal{D}$-module arises in this way, and there is a bijection given by restriction between the simple $R\mathcal{C}$-modules which are non-zero on $\mathcal{D}$, and the simple $R\mathcal{D}$-modules.*

Proof. (1) Since $R\mathcal{D}\uparrow^{\mathcal{C}}_{\mathcal{D}}= R\mathcal{C}\otimes_{R\mathcal{D}} R\mathcal{D}\cong R\mathcal{C}$ is projective it follows that the induction of an arbitrary projective is projective. The property of generation also follows from the tensor product description of induction since if A is a subset of M for which $M = R\mathcal{D}\cdot A$ then

$$R\mathcal{C}(1_{R\mathcal{D}}\otimes_{R\mathcal{D}} A) = R\mathcal{C}\otimes_{R\mathcal{D}} (R\mathcal{D}\cdot A) = R\mathcal{C}\otimes_{R\mathcal{D}} M = M\uparrow^{\mathcal{C}}_{\mathcal{D}}.$$

We have

$$M\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}}= 1_{R\mathcal{D}}(R\mathcal{C}\otimes_{R\mathcal{D}} M) = 1_{R\mathcal{D}}R\mathcal{C}1_{R\mathcal{D}}\otimes_{R\mathcal{D}} M = R\mathcal{D}\otimes_{R\mathcal{D}} M \cong M.$$

(2) Exactness of a sequence of functors is detected by evaluating the functors at objects and if the evaluations are exact on all objects of $\mathcal{C}$, they are also exact on all objects of $\mathcal{D}$. If T is a simple $R\mathcal{C}$-module and $x = 1_{R\mathcal{D}}x$ any non-zero element of $T\downarrow^{\mathcal{C}}_{\mathcal{D}}$ then $T = R\mathcal{C}1_{R\mathcal{D}}x$ by simplicity so $T\downarrow^{\mathcal{C}}_{\mathcal{D}}= 1_{R\mathcal{D}}R\mathcal{C}1_{R\mathcal{D}}x = R\mathcal{D}x$, from which it follows that $T\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is simple since it is generated by any non-zero element.

Now let S be a simple $R\mathcal{D}$-module, and consider $N = \{x \in S\uparrow^{\mathcal{C}}_{\mathcal{D}} \mid 1_{R\mathcal{D}}R\mathcal{C}x = 0\}$. This is the largest $R\mathcal{C}$-submodule of $S\uparrow^{\mathcal{C}}_{\mathcal{D}}$ which is zero on $\mathcal{D}$. Observe that $S\uparrow^{\mathcal{C}}_{\mathcal{D}}$ is generated by any element which is non-zero on $\mathcal{D}$, since such an element generates the restriction to $\mathcal{D}$ by simplicity of S, and this generates $S\uparrow^{\mathcal{C}}_{\mathcal{D}}$ by part (1). It follows that

N is the unique maximal submodule of $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$, since if $x\notin N$ then $R\mathcal{C}x = S\uparrow_{\mathcal{D}}^{\mathcal{C}}$ as just observed. In particular $\hat{S} := S\uparrow_{\mathcal{D}}^{\mathcal{C}}/N$ is a simple $R\mathcal{C}$-module. Since $N\downarrow_{\mathcal{D}}^{\mathcal{C}}= 0$ we have $\hat{S}\downarrow_{\mathcal{D}}^{\mathcal{C}}= S\uparrow_{\mathcal{D}}^{\mathcal{C}}\downarrow_{\mathcal{D}}^{\mathcal{C}}= S$ by part (1). This shows that every simple $R\mathcal{D}$-module arises as the restriction of a simple $R\mathcal{C}$-module. If T is a simple $R\mathcal{C}$-module for which $T\downarrow_{\mathcal{D}}^{\mathcal{C}}\cong S$ then by adjointness we have a non-zero homomorphism $S\uparrow_{\mathcal{D}}^{\mathcal{C}}\to T$ from which it follows that T is a simple quotient of $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$, and hence $T\cong\hat{S}$. This completes the proof. □

It is a remarkable and useful property that every simple module defined on $\mathcal{D}$ extends to a simple module defined on $\mathcal{C}$. More can be said about this relationship, and we mention that a theory of relative projectivity, vertices and sources inspired by Green's theory for group representations is developed in the thesis of Xu [Xu]. Reference needed

6.3 Parametrization of simple and projective representations

We start by parametrizing the simple representations of a category $\mathcal{C}$. It is the case that they are naturally defined over a field R, and we could make the assumption that R is a field without loss of generality if we wish. In fact it does not seem to make a difference to the first results of this section.

We start by repeating Proposition 6.2.3 in a special case.

Proposition 6.3.1. *Let S be a simple representation of $\mathcal{C}$ over R.*

1. *For every full subcategory $\mathcal{D}$ of $\mathcal{C}$ with finitely many objects the restriction $S\downarrow_{\mathcal{D}}^{\mathcal{C}}$ is either a simple $R\mathcal{D}$-module or zero.*

2. *For every object x of $\mathcal{C}$ the evaluation $S(x)$ is a simple $R\operatorname{End}_{\mathcal{C}}(x)$-module.*

3. *If T is another simple representation of $\mathcal{C}$ over R and x is an object of $\mathcal{C}$ for which $T(x)\cong S(x)$ as $R\operatorname{End}_{\mathcal{C}}(x)$-modules, and $T(x)\neq 0$, then $S\cong T$ as representations of $\mathcal{C}$.*

Proof. The result merely restates and interprets Proposition 3.2, and (1) is nothing more than this. For (2) and (3) we apply Proposition 3.2 in the case of the full subcategory which has x as its only object. Here the category algebra is $R\operatorname{End}_{\mathcal{C}}(x)$ and the statements follow immediately. □

Consider the set of pairs (x,V) where x is an object of $\mathcal{C}$ and V is a simple $R\operatorname{End}_{\mathcal{C}}(x)$-module. We will write $(x,V)\sim(y,W)$ if and only if there is a simple $R\mathcal{C}$-module S with $S(x)\cong V$ and $S(y)\cong W$. Certainly if x and y are isomorphic in $\mathcal{C}$ and $V\cong W$ as $R\operatorname{End}_{\mathcal{C}}(x)$-modules, where the action of $\operatorname{End}_{\mathcal{C}}(x)$ on W is transported via an isomorphism between x and y, then $(x,V)\sim(y,W)$, but this property may arise in other circumstances as well, as we will illustrate by example after the next result, which follows immediately from Proposition 6.3.1.

Corollary 6.3.2. *1. The relation $\sim$ is an equivalence relation on the set of pairs (x, V) where x ranges through objects of $\mathcal{C}$ and V ranges through simple $R\,\mathrm{End}_{\mathcal{C}}(x)$-modules.*

2. The isomorphism classes of simple representations of $\mathcal{C}$ are in bijection with the equivalence classes of pairs (x, V), the bijection sending a simple module S to the equivalence class of $(x, S(x))$, where x is any object of $\mathcal{C}$ for which $S(x) \neq 0$.

Example 6.3.3. Let $\mathcal{C}$ be the category with two objects, x and y, and with morphisms 1_x, 1_y, $\alpha : x \to y$, $\beta : y \to x$ and $\gamma : y \to y$, satisfying $\beta\alpha = 1_x$ and $\alpha\beta = \gamma$. From this it follows that $\gamma^2 = \gamma$, $\beta\gamma = \beta$ and $\gamma\alpha = \alpha$.

$$\begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, bend left, "\alpha"] & y \arrow[l, bend left, "\beta"] \arrow[loop above, "1_y"] \arrow[loop below, "\gamma"] \end{tikzcd}$$

This category may be identified as the full subcategory of the category of $\mathbb{F}_2$-vector spaces whose objects are the vector spaces of dimensions 0 and 1, and it was already considered in Example 6.2.2.

We see that $\mathbb{Q}\,\mathrm{End}_{\mathcal{C}}(x) = \mathbb{Q}$ has one simple module and

$$\mathbb{Q}\,\mathrm{End}_{\mathcal{C}}(y) \cong \mathbb{Q}[c]/(c^2 - c) \cong \mathbb{Q}[c]/(c) \oplus \mathbb{Q}[c]/(c-1) \cong \mathbb{Q} \oplus \mathbb{Q}$$

has two simple modules, giving rise to pairs $(x, \mathbb{Q}), (y, \mathbb{Q}_0), (y, \mathbb{Q}_1)$, where γ acts on $\mathbb{Q}_0$ and $\mathbb{Q}_1$ as multiplication by 0 and 1, respectively. We see that

$$\mathbb{Q}\mathcal{C} = \mathbb{Q}\langle 1_x, \alpha\rangle \oplus \mathbb{Q}\langle \gamma, \beta\rangle \oplus \mathbb{Q}\langle 1_y - \gamma\rangle$$

as $\mathbb{Q}\mathcal{C}$-modules, and that the three submodules in the decomposition are simple and $\mathbb{Q}\langle 1_x, \alpha\rangle \cong \mathbb{Q}\langle \gamma, \beta\rangle$. In fact the first two modules, when regarded as functors, both take the value $\mathbb{Q}$ on x and y, and every morphism acts as the identity morphism – they are the constant functor. They are simple because they are generated by any non-zero vector which they contain. Thus $\mathbb{Q}\mathcal{C}$ has two simple modules, $S_{x,\mathbb{Q}} \cong S_{y,\mathbb{Q}_1}$ and $S_{y,\mathbb{Q}_0}$. Under the equivalence relation $\sim$ the equivalence classes are $\{(x, \mathbb{Q}), (y, \mathbb{Q}_1)\}$ and $\{(y, \mathbb{Q}_0)\}$.

Exercise: Show that the category of representations of vector spaces of dimension at most n is equivalent to the category of representations of the monoid Mat_n.

Theorem 6.3.4 (Kuhn)**.** *Let $M \in \mathrm{Rep}(F\text{-mod}, F)$ be simple, and let n be minimal such that $M(F^n) \neq 0$. Then $M(F^n)$ is a simple $GL_n(F)$-module and singular $n \times n$-matrices act as zero on it. The pair $(n, M(F^n)$ completely determines M, and for each such pair there is a simple functor M realizing the pair. Thus isomorphism classes of simple functors biject with $\bigcup_{j\geq 0}\{$simple $GL_j(F)$-modules$\}$.*

Proof. This is immediate from Proposition 6.3.1. Note that if M is a simple functor then all its evaluations $M(F^m)$ are simple or zero as modules for the monoid Mat_m. By minimality of n and the fact that singular matrices factor through F^{n-1}, such matrices act as zero on $M(F^n)$, because $M(F^{n-1})$ is zero. Thus $M(F^n)$ is simple as a representation of $GL_n(F)$, by Proposition 6.1.4. $\square$

To verify in practice that a particular functor is simple the following criterion can be useful.

Theorem 6.3.5. *Let M be a functor in* $\mathrm{Rep}(F\text{-}mod, F)$ *and let n be minimal with $M(F^n) \neq 0$. Then M is simple if and only if*

1. *$M(F^n)$ is simple as a $GL_n(F)$-module,*
2. *for all natural numbers j the sum of the images $M(\alpha) : M(F^n) \to M(F^j)$, taken over all linear maps $\alpha : F^n \to F^j$, equals $M(F^j)$, and*
3. *for all natural numbers j the intersection of the kernels of maps $M(\beta) : M(F^j) \to M(F^n)$, taken over all linear maps $\beta : F^j \to F^n$, is zero.*

Proof. Observe that for each natural number j, putting $U(F^j)$ to be the sum of the images of the $M(\alpha)$ as in part (2), defines a subfunctor U of M, and putting $V(F^j)$ to be the intersection of the kernels of the $M(\beta)$ as in part (3), also defines a subfunctor V of M.

If M is simple we have already seen that $M(F^n)$ is a simple module. Furthermore, U and V must be zero or M, and since $U(F^n) = M(F^n) \neq 0$ we have $U = M$, and since $V(F^n) = 0 \neq M(F^n)$ we deduce $V = 0$.

Conversely, suppose the three conditions hold and suppose that N is a nonzero subfunctor of M. By condition (3) there is a linear map $\beta : F^j \to F^n$ for some j so that $M(\beta)$ is nonzero on $N(F^j)$. Thus $N(F^n) \neq 0$, and this is a nonzero $\mathrm{End}\, F^n$-submodule of $M(F^n)$, which is simple. It follows that $N(F^n) = M(F^n)$. By condition (2), which says that M is generated by its value at F^n, we have $M(F^j) = N(F^j)$ for all j, so that $M = N$. This shows that M is a simple functor. $\square$

We present an application of this theorem.

Corollary 6.3.6. *For all n the exterior power functor Λ^n is simple.*

Proof. We have seen before that every $\Lambda^n(F^j)$ is either simple as a $GL(F^j)$-module, or zero, and from this a proof can be readily constructed. We may also prove it using Theorem 6.3.5. We verify the three conditions of that theorem. The minimal index j so that $\Lambda^n(F^j) \neq 0$ is $j = n$, and $\Lambda^n(F^n)$ has dimension 1, so it is a simple module. This shows that condition (1) of the theorem holds. If $1 \leq i_1 < \cdots < i_n \leq j$ then the linear map $\alpha : F^n \to F^j$ defined by $\alpha(e_k) = e_{i_k}$ induces a map $\Lambda(F^n) \to \Lambda(F^j)$ mapping onto the span of $e_{i_1} \wedge \cdots \wedge e_{i_n}$. Since such vectors span $\Lambda(F^j)$ we see that condition (2) is satisfied. To verify (3), suppose that $\sum c_{i_1 \ldots i_n}(e_{i_1} \wedge \cdots \wedge e_{i_n})$ is an element of $\Lambda^n(F^j)$, and suppose that some particular coefficient $c_{i_1 \ldots i_n}$ is nonzero. The map $\beta : F^j \to F^n$ specified by $\beta(e_{i_k}) = e_k$, and $\beta(e_i) = 0$ if $i \notin \{i_1, \ldots, i_n\}$, sends the sum to $c_{i_1 \ldots i_n}(e_1 \wedge \cdots \wedge e_n)$, which is nonzero. This shows that no nonzero element of $\Lambda^n(F^j)$ is mapped to zero by all linear maps $F^j \to F^n$, provided $j \geq n$, thus showing that condition (3) of Theorem 6.3.5 holds. $\square$

To do: continue to show that $\Lambda^n(F^j)$ is simple as a representation of $GL_j(F)$.

Corollary 6.3.7. *For every $j \geq n$ the $\mathrm{Mat}_j(F)$-representation $\Lambda^n(F^j)$ is simple.*

Example 6.3.8. We identify Λ^n as the simple functor $S_{n,\det}$ where det denotes the one-dimensional representation of $GL_n(F)$ on $\Lambda^n(F^n)$. When $n = 0$ this gives the constant functor $\Lambda^0 = S_{0,F}$. The table of dimensions of evaluations of these functors is Pascal's triangle.

6.4 Projective functors

This next material is adapted from Webb: Introduction to representations and cohomology of categories.

We now turn to the projective representations of a category $\mathcal{C}$, and for this we will assume that R is a field or a complete discrete valuation ring. When $\mathcal{C}$ is finite, $R\mathcal{C}$ is an R-algebra of finite rank, the Krull-Schmidt theorem holds, and each simple representation $S_{x,V}$ has a projective cover $P_{x,V}$. This gives a parametrization of the indecomposable projective representations by the equivalence classes of pairs (x, V) in this case.

The category of finite dimensional vector spaces does not have finitely many objects, but its projective representations behave in the same way as if it did have finitely many objects. For a general category $\mathcal{C}$, for each object $x \in \mathrm{Ob}(\mathcal{C})$ we may construct a *linearized representable functor* $P_x : \mathcal{C} \to R$-mod defined by $P_x(y) = R\,\mathrm{Hom}_{\mathcal{C}}(x, y)$, the free R-module with the elements of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ as a basis.

Class exercise: figure out some entries of the table without it being given.

Example 6.4.1. When $F = \mathbb{F}_p$ is the field of p elements and $\mathcal{C} = F$-mod the dimensions of the evaluations of representable functors $P_x(F^n)$ are given in the following table. Note that $P_F(F^n)$ may be identified with the group ring of the elementary abelian group F^n.

Dimensions of evaluations of representable functors

	P_0	P_F	$\cdots$	P_{F^j}
F^0	1	1		1
F^1	1	p		p^j
F^2	1	p^2		p^{2j}
$\vdots$				
F^n	1	p^n		p^{nj}

The following identification of representable functors on the category of vector spaces is intriguing, although we will not use it.

Proposition 6.4.2. *Let $\mathcal{C}$ be the category of finite dimensional vector spaces over F. If V_1 and V_2 are vector spaces then $P_{V_1 \oplus V_2} \cong P_{V_1} \otimes_R P_{V_2}$. In particular, $P_{F^n} \cong (P_F)^{\otimes n}$.*

Proof. It follows from

$$\begin{aligned} P_{V_1 \oplus V_2}(W) &= R\,\mathrm{Hom}(V_1 \oplus V_2, W) \\ &\cong R[\mathrm{Hom}(V_1, W) \times \mathrm{Hom}(V_2, W)] \\ &\cong R\,\mathrm{Hom}(V_1, W) \otimes_R R\,\mathrm{Hom}(V_2, W) \\ &= (P_{V_1} \otimes_R P_{V_2})(W). \end{aligned}$$

□

Proposition 6.4.3. *Let x be an object of a category $\mathcal{C}$.*

1. *(Yoneda's lemma) Let M be a representation of $\mathcal{C}$. Then $\mathrm{Hom}_{R\mathcal{C}}(P_x, M) \cong M(x)$.*
2. *The representation P_x is projective and generated by its value at x.*
3. *Regarded as an $R\mathcal{C}$-module, $P_x \cong R\mathcal{C}1_x$.*
4. *Let $\mathcal{D}$ be any full subcategory of $\mathcal{C}$ which contains x. Then $P_x \cong P_x^{\mathcal{D}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ where $P_x^{\mathcal{D}} = P_x \downarrow_{\mathcal{D}}^{\mathcal{C}}$ is the functor P_x constructed for $\mathcal{D}$.*

Proof. (1) We define $\alpha : \mathrm{Hom}_{R\mathcal{C}}(P_x, M) \to M(x)$ by $\alpha(\eta) = \eta_x(1_x)$. In the opposite direction we define $\beta : M(x) \to \mathrm{Hom}_{R\mathcal{C}}(P_x, M)$ as follows: if $u \in M(x)$ and $\sum_{\gamma:x\to y} \lambda_\gamma \gamma \in P_x(y) = R\,\mathrm{Hom}_{\mathcal{C}}(x, y)$ we put $\beta(u)_y(\sum_{\gamma:x\to y} \lambda_\gamma\gamma) = \sum_{\gamma:x\to y} \lambda_\gamma M(\gamma)(u)$. We verify in the usual way that α and β are mutually inverse isomorphisms.

(2) Suppose we have an epimorphism of representations $\theta : M \to N$ and a morphism $\eta : P_x \to N$. We may find $u \in M(x)$ so that $\theta(u) = \eta_x(1_x)$. Now the morphism $\beta(u) : P_x \to M$ satisfies $\theta \circ \beta(u) = \eta$ since $\theta_x(\beta(u)_x(1_x)) = \theta_x(u) = \eta_x(1_x)$. If $\gamma : x \to y$ then $\gamma = P_x(\gamma)(1_x)$ lies in the subfunctor of P_x generated by $1_x \in P_x(x)$. This shows that P_x is generated by its value at x.

(3) From the definitions, the value of $R\mathcal{C}1_x$ at an object y is

$$1_y R\mathcal{C}1_x = R\,\mathrm{Hom}_{\mathcal{C}}(x, y) = P_x(y)$$

and this shows that $P_x \cong R\mathcal{C}1_x$ as $R\mathcal{C}$-modules.

(4) We have $P_x^{\mathcal{D}} = R\mathcal{D}1_x$ so

$$P_x^{\mathcal{D}} \uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} R\mathcal{D}1_x = R\mathcal{C}1_x = P_x.$$

Also, if y is an object of $\mathcal{D}$ then

$$P_x \downarrow_{\mathcal{D}}^{\mathcal{C}} (y) = 1_y R\mathcal{C}1_x = \mathrm{Hom}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{D}}(x, y) = P_x^{\mathcal{D}}(y),$$

which shows that $P_x^{\mathcal{D}} = P_x \downarrow_{\mathcal{D}}^{\mathcal{C}}$. □

This corollary needs to be adapted to the categories being considered. See below.

Corollary 6.4.4. *Suppose that R is a field or a complete discrete valuation ring and let P be an indecomposable projective $R\mathcal{C}$-module where $\mathcal{C}$ is a finite category. Then for some object x of $\mathcal{C}$, P is generated by its value at x and is isomorphic to a direct summand of P_x. It has the form $P \cong R\mathcal{C}e$ where e is a primitive idempotent in the monoid algebra $R\,\mathrm{End}_{\mathcal{C}}(x)$. Every primitive idempotent in $R\,\mathrm{End}_{\mathcal{C}}(x)$ remains primitive in $R\mathcal{C}$.*

Proof. By Yoneda's lemma, for each object y and element of $P(y)$ there is a homomorphism $F_y \to P$ having that element in its image and so P is a homomorphic image of a direct sum of representable functors $\bigoplus_i F_{x_i}$. Since P is indecomposable projective, the surjection must split and P is isomorphic to a direct summand of a functor P_x. Since P_x is generated by its value at x, so is P. Since by Yoneda's lemma $\mathrm{End}_{R\mathcal{C}}(P_x) \cong R\,\mathrm{End}_{\mathcal{C}}(x)$, the direct summand has the form $R\mathcal{C}1_x e = R\mathcal{C}e$ for some idempotent $e \in R\,\mathrm{End}_{\mathcal{C}}(x)$, and e is primitive in $R\mathcal{C}$ since P is indecomposable, hence *a fortiori* primitive in $R\,\mathrm{End}_{\mathcal{C}}(x)$.

Equally, if f is a primitive idempotent in $R\,\mathrm{End}_{\mathcal{C}}(x)$ then $R\mathcal{C}f$ is a projective $R\mathcal{C}$-module which is generated by its value at x. If $R\mathcal{C}f = M \oplus N$ is a decomposition as a direct sum of $R\mathcal{C}$-modules, then each of M and N is also generated by its value at x, and so if they are non-zero then $1_x R\mathcal{C}f = 1_x M \oplus 1_x N$ is a non-zero decomposition of the indecomposable $R\,\mathrm{End}_{\mathcal{C}}(x)$-module $R\,\mathrm{End}_{\mathcal{C}}(x)f$. Since this is not possible we deduce that $R\mathcal{C}f$ is indecomposable, and so f is primitive in $R\mathcal{C}$. □

The above result shows that the indecomposable projective $R\mathcal{C}$-modules may be parametrized by a subset of the primitive idempotents of the $R\,\mathrm{End}_{\mathcal{C}}(x)$ as x ranges over the objects of $\mathcal{C}$, and these in turn are parametrized by the equivalence classes of pairs (x, V) where x is an object of $\mathcal{C}$ and V is a simple $R\,\mathrm{End}_{\mathcal{C}}(x)$-module, since we have already seen that these parametrize the simple representations.

Corollary 6.4.5. *Suppose that R is a field or a complete discrete valuation ring and let $\mathcal{C}$ be a category for which each endomorphism monoid $\mathrm{End}_{\mathcal{C}}(x)$ is finite, as x ranges over objects of $\mathcal{C}$.*

1. $\mathrm{End}_{R\mathcal{C}}(P_x) \cong R\,\mathrm{End}_{\mathcal{C}}(x)$, *an algebra of finite rank over R.*
2. *Finitely generated projective representations satisfy the conclusion of the Krull-Schmidt theorem.*
3. *Every finitely generated indecomposable projective representation P is generated by its value at a single object x and is a direct summand of P_x.*
4. *Finitely generated representations have projective covers.*
5. *Each finitely generated indecomposable projective representation has a unique simple quotient, and is the projective cover of that simple quotient.*

Proof. To be supplied. □

Corollary 6.4.6. *Let $\mathcal{C}$ be the category of finite dimensional vector spaces over a finite field F.*

1. *The indecomposable projective representations are the projective covers $P_{n,V}$ of the simple representations $S_{n,V}$, where V is a simple $RGL_n(F)$-module.*
2. $P_{F^n} \cong \bigoplus_{n,V} P_{j,V}^{d^n_{j,V}}$, *where* $d^n_{j,V} = \dim S_{j,V}(F^n)/\dim \mathrm{End}(S_{j,V})$

In part (2) above we have that $\mathrm{End}(S_{j,V}) \cong \mathrm{End}(V)$, and it is a fact that simple representations of $GL_j(F)$ over F are absolutely simple, so that this dimension is 1. Note also that, since $S_{j,V}(F(n)) = 0$ unless $n \geq j$, the summands $P_{j,V}$ of F_{F^n} that appear with nonzero multiplicity all have $j \leq n$.

Proof. To be supplied. □

Example 6.4.7. According to the formula for the decomposition of P_{F^n} we have $P_0 = P_{0,F} = S_{0,F}$, using also the fact that $P_{0,F}$ is the projective cover of $S_{0,F}$, and that both of these functors have evaluations all of dimension 1. Thus the constant functor $S_{0,F}$ is projective.

Proposition 6.4.8. *Let $F = \mathbb{F}_2$ be the field of 2 elements. The indecomposable projective $P_{1,F}$ has as its composition factors all of the simple functors Λ^n, $n \geq 1$, with each simple functor appearing just once.*

Proof. We have $P_F = P_{0,F} \oplus P_{1,F}$ from the formula, and $P_{1,F}$ is the projective cover of $S_{1,F} = \Lambda^1$, the identity functor. We illustrate the dimensions of the evaluations of these functors in a table.

Dimensions of some evaluations

	$P_0 = \Lambda^0$	$P_F = P_{0,F} \oplus P_{1,F}$	$P_{1,F}$	$S_{1,F}$	$\mathrm{Rad}\, P_{1,F}$
F^0	1	1	0	0	0
F^1	1	2	1	1	0
F^2	1	4	3	2	1
F^3	1	8	7	3	4
F^4	1	16	15	4	11

We see that $\dim P_{1,F}(F^n) = 2^n - 1$. The only simple functor that is 0 on F^0 and of dimension 1 on F is $S_{1,F}$, so we deduce that $S_{1,F}$ is a composition factor of $P_{1,F}$ (and in any case we already knew this). We deduce that the remaining composition factors of $P_{1,F}$ have dimensions summing to 1 on F^2 and 0 on F^0 and F^1. We now repeat the argument: the only simple that is 0 on F^0 and F^1 and of dimension 1 on F^2 is $S_{2,F} = \Lambda^2$, so this must be a composition factor. We now find that the remaining composition factors have dimensions summing to 1 on F^3, and 0 on smaller spaces. We repeat the argument. Since

$$\dim P_{1,F}(F^n) = 2^n - 1 = 1 + \sum_{i=1}^{n-1} \binom{n}{i} = 1 + \sum_{i=1}^{n-1} \dim \Lambda^i(F^n)$$

we see that after subtracting the dimensions of simple composition factors Λ^i for $1 \leq i \leq n-1$ we are left with an evaluation at F^n of dimension 1, 0 on smaller spaces, forcing a composition factor Λ^n. This shows that every Λ^n appears as a composition factor with $n \geq 1$, and there are no more since the dimensions of evaluations of these functors sum to give the dimension of $P_{1,F}$ at each evaluation. □

To do: Examples over $\mathbb{F}_p$ where $p \neq 2$. Give a proof that simple functors have simple evaluations as modules for GL_n. Apply the theory of simple modules for monoids in terms of idempotents (see Linckelmann-Stolarz). Quasi-hereditary structure. Relations between weight description of simples and different simple evaluations of a simple functor. Show that simple functors are not zero above their minimal nonzero value (exercise?). Describe the application to the Steenrod algebra as endomorphism ring of the sum of the symmetric powers. The artinian conjecture of Schwartz (theorem of Sam and Snowden).

Chapter 7

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