Solutions for Chapter 1

1.1.1 a. $\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$ b. $2\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}4\\8\end{bmatrix}$

c. $\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1-2\\3-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ d. $\begin{bmatrix}3\\2\end{bmatrix}+\vec{\mathbf{e}}_1=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

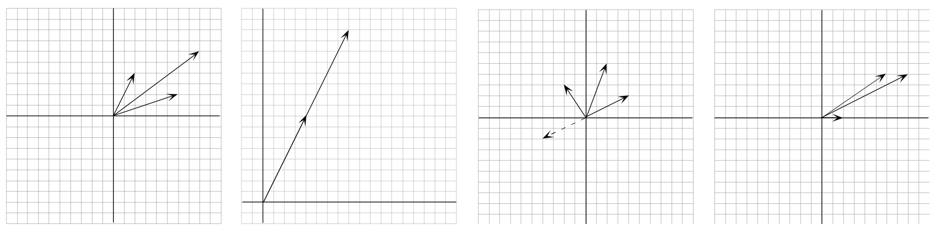


Figure for solution 1.1.1. From left: (a), (b), (c), and (d).

1.1.2

a. $\begin{bmatrix}3\\\pi\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\\sqrt{2}\end{bmatrix}=\begin{bmatrix}4\\\pi-1\\1+\sqrt{2}\end{bmatrix}$ b. $\begin{bmatrix}1\\4\\c\\2\end{bmatrix}+\vec{\mathbf{e}}_2=\begin{bmatrix}1\\5\\c\\2\end{bmatrix}$ c. $\begin{bmatrix}1\\4\\c\\2\end{bmatrix}-\vec{\mathbf{e}}_4=\begin{bmatrix}1\\4\\c\\1\end{bmatrix}$

1.1.3 a. $\vec{\mathbf{v}}\in\mathbb{R}^3$ b. $L\subset\mathbb{R}^2$ c. $C\subset\mathbb{R}^3$ d. $\mathbf{x}\in\mathbb{C}^2$,
e. $B_0\subset B_1\subset B_2,\dots$.

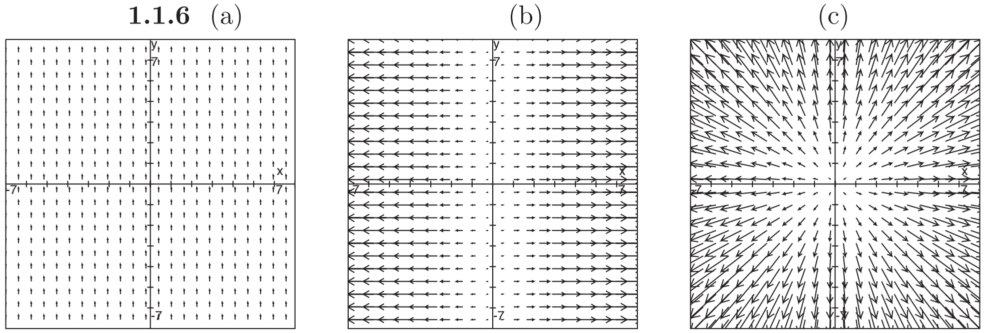
1.1.4 a. The two trivial subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$ and $\mathbb{R}^n$.
b. Yes there are. For example,

$$\begin{pmatrix}\cos\pi/6\\\sin\pi/6\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\cos\pi/3\\\sin\pi/3\end{pmatrix}.$$

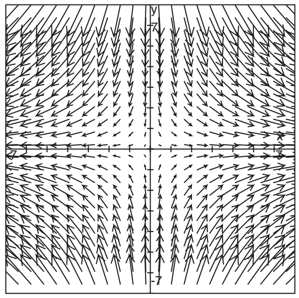
(Rotating all the vectors by any angle gives all the examples.)

1.1.5 a. $\sum_{i=1}^n\vec{\mathbf{e}}_i$ b. $\sum_{i=1}^n i\vec{\mathbf{e}}_i$ c. $\sum_{i=3}^n i\vec{\mathbf{e}}_i$

1.1.6 (a) (b) (c)



(d)



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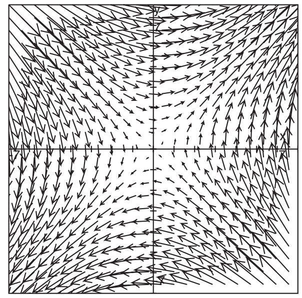
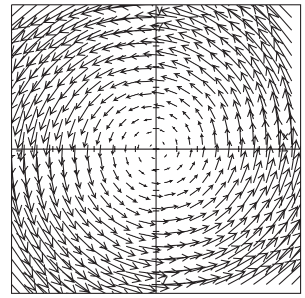
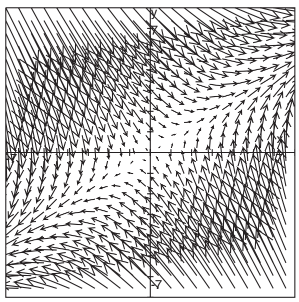


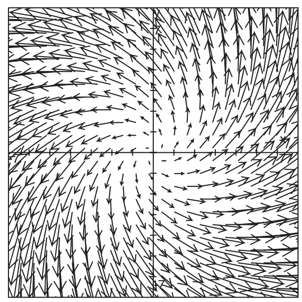
(f)



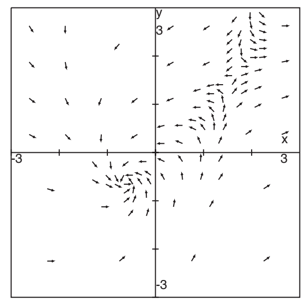
(g)



(h)

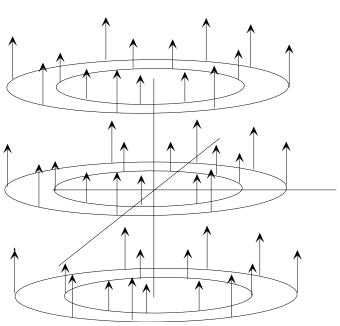


(i)

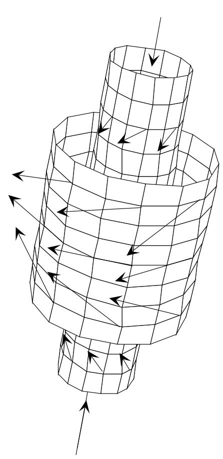


1.1.7 The vector field in part a points straight up everywhere. Its length depends only on how far you are from the z-axis, and it gets longer and longer the further you get from the z-axis; it vanishes on the z-axis. The vector field in part b is simply rotation in the (x, y)-plane, like (f) in exercise 1.1.6. But the z-component is down when $z > 0$ and up when $z < 0$. The vector field in part c spirals out in the (x, y)-plane, like (h) in exercise 1.1.6. Again, the z-component is down when $z > 0$ and up when $z < 0$.

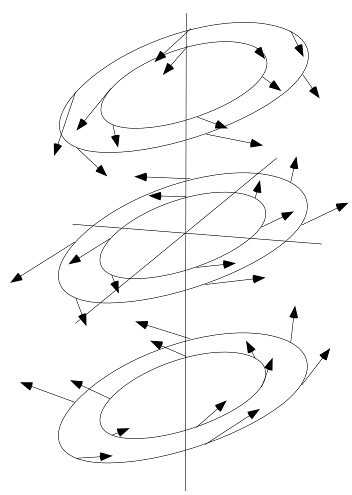
a.



b.



c.



1.1.8 (a) $\begin{bmatrix} 0 \\ 0 \\ a^2 - x^2 - y^2 \end{bmatrix}$ (b) Assuming that $a \leq 1$, flow is in the counter-clockwise direction and using cylindrical coordinates (r, θ, z) we get $\begin{bmatrix} 0 \\ (a^2 - (1-r)^2)/r \\ 0 \end{bmatrix}$

1.2.1 i. 2×3 ii. 2×2 iii. 3×2 iv. 3×4 v. 3×3

b. The matrices i and v can be multiplied on the right by the matrices iii, iv, v; the matrices ii and iii on the right by the matrices i and ii.

1.2.2

a. $\begin{bmatrix} 28 & 14 \\ 79 & 44 \end{bmatrix}$ b. impossible c. $\begin{bmatrix} 3 & 0 & -5 \\ 4 & -1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 31 \\ -5 \\ -2 \end{bmatrix}$

e. $\begin{bmatrix} -1 & 10 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 29 \\ -9 & 24 \end{bmatrix}$ f. impossible

1.2.3 a. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ b. $[6 \quad 16 \quad 2]$

1.2.4 a. This is the second column vector of the left matrix: $\begin{bmatrix} 2 \\ 8 \\ \sqrt{5} \end{bmatrix}$

b. Again, this is the second column vector of the left matrix: $\begin{bmatrix} 2 \\ 2\sqrt{a} \\ 12 \end{bmatrix}$

c. This is the third column vector of the left matrix: $\begin{bmatrix} 8 \\ \sqrt{3} \end{bmatrix}$

1.2.5 a. True: $(AB)^\top = B^\top A^\top = B^\top A$

b. True: $(A^\top B)^\top = B^\top (A^\top)^\top = B^\top A = B^\top A^\top$

c. False: $(A^\top B)^\top = B^\top (A^\top)^\top = B^\top A \neq BA$

d. False: $(AB)^\top = B^\top A^\top \neq A^\top B^\top$

1.2.6 Diagonal: (a), (b), (d), and (g)

Symmetric: (a), (b), (d), (g), (h), (j)

Triangular: (a), (b), (c), (d),(e), (f), (g), (i), and (l)

No antisymmetric matrices

Results of multiplications: b. $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$

c. $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ab & ab \end{bmatrix}$

d. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ e. $\begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ a^2 & a^2 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 \\ a^3 & a^3 \end{bmatrix}$ g. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ j. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ l. $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

1.2.7 The matrices a and d have no transposes here. The matrices b and f are transposes of each other. The matrices c and e are transposes of each other.

1.2.8
$$AB = \begin{bmatrix} 1+a & 1 \\ 1 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 1 \\ 1+a & a \end{bmatrix}$$

So $AB = BA$ only if $a = 0$.

1.2.9 a. $A^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B^\top = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ b. $(AB)^\top = B^\top A^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

c. $(AB)^\top = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

d. The matrix multiplication $A^\top B^\top$ is impossible.

1.2.10
$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^{-1} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}, \qquad \text{which exists when } a \neq 0.$$

1.2.11 The expressions b, c, d, f, g, and i make no sense.

1.2.12
$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.2.13 The trivial case is when $a = b = c = d = 0$; then obviously $ad - bc = 0$ and the matrix is not invertible. Let us suppose $d \neq 0$. (If we suppose that any other entry is nonzero, the proof would work the same way.) If $ad = bc$, then the first row is a multiple of the second: we can write $a = \frac{b}{d}c$ and $b = \frac{b}{d}d$, so the matrix is $A = \begin{bmatrix} \frac{b}{d}c & \frac{b}{d}d \\ c & d \end{bmatrix}$.

To show that A is not invertible, we need to show that there is no matrix $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But if the upper left corner of AB is 1, then we have $\frac{b}{d}(a'c + c'd) = 1$, so the lower left corner, which is $a'c + c'd$, cannot be 0.

1.2.14 Let $C = AB$. Then

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j},$$

so if $D = C^\top$ then

$$d_{i,j} = c_{j,i} = \sum_{k=1}^{n} a_{j,k} b_{k,i}.$$

Let $E = B^\top A^\top$. Then

$$e_{i,j} = \sum_{k=1}^{n} b_{k,i} a_{j,k} = \sum_{k=1}^{n} a_{j,k} b_{k,i} = d_{i,j},$$

so $E = D$. So $(AB)^\top = B^\top A^\top$.

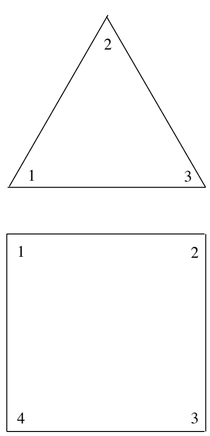
1.2.15

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & az+b+y \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix}$$

So $x = -a$, $z = -c$ and $y = ac - b$.

1.2.16 This is a straightforward computation, using $(AB)^\top = B^\top A^\top$:

$$(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A.$$



Labeling for solution 1.2.17

1.2.17 With the labeling shown in the margin, the adjacency matrices are

a. $$A_T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

b. $$A_T^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad A_T^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \quad A_T^4 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$$

$$A_T^5 = \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix}$$

$$A_S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad A_S^3 = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} \quad A_S^4 = \begin{bmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{bmatrix}$$

$$A_S^5 = \begin{bmatrix} 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \\ 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \end{bmatrix}$$

The diagonal entries of A^n are the number of walks we can take of length n that take us back to our starting point.

c. In a triangle, by symmetry there are only two different numbers: the number a_n of walks of length n from a vertex to itself, and the number b_n of walks of length n from a vertex to a different vertex. The recurrence relation relating these is

$$a_{n+1} = 2b_n \quad \text{and} \quad b_{n+1} = a_n + b_n.$$

These reflect that to walk from a vertex V_1 to itself in time $n+1$, at time n we must be at either V_2 or V_3, but to walk from a vertex V_1 to a different vertex V_2 in time $n+1$, at time n we must be either at V_1 or at V_3. If $|a_n - b_n| = 1$, then $a_{n+1} - b_{n+1} = |2b_2 - (a_n + b_n)| = |b_n - a_n| = 1$.

d. Color two opposite vertices of the square black and the other two white. Every move takes you from a vertex to a vertex of the opposite color. Thus if you start at time 0 on black, you will be on black at all even times, and on white at all odd times, and there will be no walks of odd length from a vertex to itself.

e. Suppose such a coloring in black and white exists; then every walk goes from black to white to black to white $\dots$, in particular the (B, B) and the (W, W) entries of A^n are 0 for all odd n, and the (B, W) and (W, B) entries are 0 for all even n. Moreover, since the graph is connected, for any pair of vertices there is a walk of some length m joining them, and then the corresponding entry is nonzero for m, $m+2$, $m+4, \dots$ since you can go from the point of departure to the point of arrival in time m, and then bounce back and forth between this vertex and one of its neighbors.

Conversely, suppose the entries of A^n are zero or nonzero as described, and look at the top line of A^n, where n is chosen sufficiently large so that any entry that is ever nonzero is nonzero for A^{n-1} or A^n. The entries correspond to pairs of vertices (V_1, V_i); color in white the vertices V_i for which the $(1, i)$ entry of A^n is zero, and in black those for which the $(1, i)$ entry of A^{n+1} is zero. By hypothesis, we have colored all the vertices. It remains to show that adjacent vertices have different colors. Take a path of length m from V_1 to V_i. If V_j is adjacent to V_i, then there certainly exists a path of length $m+1$ from V_1 to V_j, namely the previous path, extended by one to go from V_i to V_j. Thus V_i and V_j have opposite colors.

1.2.18

(a) $$A^2 = \begin{bmatrix} 3 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 3 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 3 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 3 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 7 & 0 & 7 & 0 & 7 & 0 & 6 \\ 7 & 0 & 7 & 0 & 6 & 0 & 7 & 0 \\ 0 & 7 & 0 & 7 & 0 & 6 & 0 & 7 \\ 7 & 0 & 7 & 0 & 7 & 0 & 6 & 0 \\ 0 & 6 & 0 & 7 & 0 & 7 & 0 & 7 \\ 7 & 0 & 6 & 0 & 7 & 0 & 7 & 0 \\ 0 & 7 & 0 & 6 & 0 & 7 & 0 & 7 \\ 6 & 0 & 7 & 0 & 7 & 0 & 7 & 0 \end{bmatrix}$$