0.7.1 "Modulus of z," "absolute value of z," and $|z|$ are synonyms. "Real part of z" is the same as $\operatorname{Re} z = a$. "Imaginary part of z" is the same as $\operatorname{Im} z = b$. The "complex conjugate of z" is the same as $\bar{z}$.

0.7.2

0.7.3 a. The absolute value of $2+4i$ is $|2+4i| = 2\sqrt{5}$. The argument (polar angle) of $2+4i$ is $\arccos 1/\sqrt{5}$, which you could also write as $\arctan 2$.

b. The absolute value of $(3+4i)^{-1}$ is $1/5$. The argument (polar angle) is $-\arccos(3/5)$.

c. The absolute value of $(1+i)^5$ is $4\sqrt{2}$. The argument is $5\pi/4$. (The complex number $1+i$ has absolute value $\sqrt{2}$ and polar angle $\pi/4$. De Moivre's formula says how to compute these for $(1+i)^5$.)

d. The absolute value of $1+4i$ is $\sqrt{17}$; the argument is $\arccos 1/\sqrt{17}$.

0.7.4 a.

$$|3+2i| = \sqrt{3^2+2^2} = \sqrt{13}; \quad \arctan\frac{2}{3} \approx .588003.$$

Remark. The angle is in radians; all angles will be in radians unless explicitly stated otherwise.

b.

$$|(1-i)^4| = |1-i|^4 = (\sqrt{2})^4 = 4; \quad \arg((1-i)^4) = 4\arg(1-i) = 4\left(-\frac{\pi}{4}\right) = -\pi.$$

One could also just observe that $(1-i)^4 = ((1-i)^2)^2 = (-i)^2 = -4$.

c.

$$|2+i| = \sqrt{5}; \quad \arg(2+i) = \arctan 1/2 \approx .463648.$$

d.

$$|\sqrt[7]{3+4i}| = \sqrt[7]{\sqrt{25}} \approx 1.2585; \quad \arg\sqrt[7]{3+4i} = \frac{1}{7}\left(\arctan\frac{4}{3} + \frac{2k\pi}{7}\right).$$

These numbers are

$$\approx .132471,\ 1.03007,\ 1.92767,\ 2.82526,\ 3.72286,\ 4.62046,\ 5.51806.$$

Remark. In this case, we have to be careful about the argument. A complex number doesn't have just one 7th root, it has seven of them, all with the same modulus but different arguments, differing by integer multiples of $2\pi/7$.

0.7.5 Parts 1–4 are immediate. For part 5, we find

$$\begin{aligned}(z_1z_2)z_3 &= \big((x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)\big)(x_3 + iy_3)\\ &= (x_1x_2x_3 - y_1y_2x_3 - y_1x_2y_3 - x_1y_2y_3)\\ &\quad + i(x_1x_2y_3 - y_1y_2y_3 + y_1x_2x_3 + x_1y_2x_3),\end{aligned}$$

which is equal to

$$\begin{aligned} z_1(z_2z_3) &= (x_1+iy_1)((x_2x_3-y_2y_3)+i(y_2x_3+x_2y_3)) \\ &= (x_1x_2x_3-x_1y_2y_3-y_1y_2x_3-y_1x_2y_3) \\ &\quad +i(y_1x_2x_3-y_1y_2y_3+x_1y_2x_3+x_1x_2y_3). \end{aligned}$$

Parts 6 and 7 are immediate. For part 8, multiply out:

$$\begin{aligned} (a+ib)\left(\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}\right) &= \frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}+i\left(\frac{ab}{a^2+b^2}-\frac{ab}{a^2+b^2}\right) \\ &= 1+i0=1. \end{aligned}$$

Part 9 is also a matter of multiplying out:

$$\begin{aligned} z_1(z_2+z_3) &= (x_1+iy_1)\big((x_2+iy_2)+(x_3+iy_3)\big) \\ &= (x_1+iy_1)((x_2+x_3)+i(y_2+y_3)) \\ &= x_1(x_2+x_3)-y_1(y_2+y_3)+i\big(y_1(x_2+x_3)+x_1(y_2+y_3)\big) \\ &= x_1x_2-y_1y_2+i(y_1x_2+x_1y_2)+x_1x_3-y_1y_3+i(y_1x_3+x_1y_3) \\ &= z_1z_2+z_1z_3. \end{aligned}$$

0.7.6 a. The quadratic formula gives

$$x=\frac{-i\pm\sqrt{i^2-4}}{2}=\frac{-i\pm\sqrt{-5}}{2}=-\frac{i}{2}(-1\pm\sqrt{5}).$$

b. The quadratic formula gives

$$x^2=\frac{-1\pm\sqrt{1-4}}{2}=\frac{1}{2}(-1\pm i\sqrt{3}).$$

These aren't any old complex numbers: they are the non-real cubic roots of 1, and their square roots are the non-real sixth roots of 1:

$$\frac{1}{2}(\pm 1\pm i\sqrt{3}).$$

Remark. You didn't have to "notice" that $(-1+i\sqrt{3})/2$ is a cubic root of 1, the square root could have been computed in the standard way anyway. Why are the solutions 6th roots of 1? Because

$$x^6-1=(x^2-1)(x^4+x^2+1),$$

so all roots of x^4+x^2+1 will also be roots of x^6-1.

Solution 0.7.7: Remember that the set of points such that the sum of their distances to two points is constant, is an ellipse, with foci at those points.

0.7.7 a. The equation $|z-u|+|z-v|=c$ represents an ellipse with foci at u and v, at least if $c>|u-v|$. If $c=|u-v|$ it is the degenerate ellipse consisting of just the segment $[u,v]$, and if $c<|u-v|$ it is empty, by the triangle inequality, which asserts that if there is a z satisfying the equality, then

$$c<|u-v|\le|u-z|+|z-v|=c.$$

b. Set $z = x + iy$; the inequality $|z| < 1 - \operatorname{Re} z$ becomes

$$\sqrt{x^2 + y^2} < 1 - x,$$

corresponding to a region bounded by the curve of equation

$$\sqrt{x^2 + y^2} = 1 - x.$$

We should worry whether the squaring introduced parasitic points, where $-\sqrt{x^2+y^2} < 1-x$, but this is not the case, since $1-x$ is positive throughout the region.

If we square this equation, we will get the curve of equation

$$x^2 + y^2 = 1 - 2x + x^2, \quad \text{i.e.,} \quad x = \frac{1}{2}(1 - y^2),$$

which is a parabola lying on its side. The original inequality corresponds to the inside of the parabola.

0.7.8

0.7.9 a. The quadratic formula gives $x = \dfrac{-i \pm \sqrt{-1-8}}{2}$, so the solutions are $x = i$ and $x = -2i$.

b. In this case, the quadratic formula gives

$$x^2 = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm i\sqrt{7}}{2}.$$

Each of these numbers has two square roots, which we still need to find.

One way, probably the best, is to use the polar form; this gives

$$x^2 = r(\cos\theta \pm i \sin\theta),$$

where

$$r = \frac{\sqrt{1+7}}{2} = \sqrt{2}, \quad \theta = \pm \arccos -\frac{1}{2\sqrt{2}} \approx 1.2094\ldots \text{ radians.}$$

Thus the four roots are

$$\pm\sqrt[4]{2}(\cos\theta/2 + i\sin\theta/2) \quad \text{and} \quad \pm\sqrt[4]{2}(\cos\theta/2 - i\sin\theta/2).$$

c. Multiplying the first equation through by $(1+i)$ and the second by i gives

$$\begin{aligned} i(1+i)x - (2+i)(1+i)y &= 3(1+i) \\ i(1+i)x - \qquad\qquad\quad y &= 4i, \end{aligned}$$

which gives

$$-(2+i)(1+i)y + y = 3 - i, \quad \text{i.e.,} \quad y = i + \frac{1}{3}.$$

Substituting this value for y then gives $x = \frac{7}{3} - \frac{8}{3}i$.

0.7.10

0.7.11 a. These are the vertical line $x = 1$ and the circle centered at the origin of radius 3.

b. Use $Z = X + iY$ as the variable in the codomain. Then

$$(1 + iy)^2 = 1 - y^2 + 2iy = X + iY$$

gives $1 - X = y^2 = Y^2/4$. Thus the image of the line is the curve of equation $X = 1 - Y^2/4$, which is a parabola with horizontal axis.

The image of the circle is another circle, centered at the origin, of radius 9, i.e., the curve of equation $X^2 + Y^2 = 81$.

c. This time use $Z = X + iY$ as the variable in the domain. Then the inverse image of the line $= \operatorname{Re} z = 1$ is the curve of equation

$$\operatorname{Re}(X + iY)^2 = X^2 - Y^2 = 1,$$

which is a hyperbola. The inverse image of the curve of equation $|z| = 3$ is the curve of equation $|Z^2| = |Z|^2 = 3$, i.e., $|Z| = \sqrt{3}$, the circle of radius $\sqrt{3}$ centered at the origin.

0.7.12

0.7.13 a. The cube roots of 1 are

$$1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

b. The fourth roots of 1 are 1, i, -1, $-i$.

c. The sixth roots of 1 are

$$1, \quad -1, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

0.7.14

0.7.15 a. The fifth roots of 1 are

$$\cos 2\pi k/5 + i\sin 2\pi k/5, \quad \text{for } k = 0, 1, 2, 3, 4.$$

The point of the question is to find these numbers in some more manageable form. One possible approach is to set $\theta = 2\pi/5$, and to observe that $\cos 4\theta = \cos\theta$. If you set $x = \cos\theta$, this leads to the equation

$$2(2x^2 - 1)^2 - 1 = x \quad \text{i.e.,} \quad 8x^4 - 8x^2 - x + 1 = 0.$$

This still isn't too manageable, until you start asking what other angles satisfy $\cos 4\theta = \cos\theta$. Of course $\theta = 0$ does, meaning that $x = 1$ is one root of our equation. But $\theta = 2\pi/3$ does also, meaning that $-1/2$ is also a root. Thus we can divide:

$$\frac{8x^4 - 8x^2 - x + 1}{(x - 1)(2x + 1)} = 4x^2 + 2x - 1,$$

and $\cos 2\pi/5$ is the positive root of that quadratic equation, i.e.,

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}, \text{ which gives } \sin\frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

The fifth roots of 1 are now

$$1, \frac{\sqrt{5}-1}{4} \pm i\frac{\sqrt{10+2\sqrt{5}}}{4}, -\frac{\sqrt{5}+1}{4} \pm i\frac{\sqrt{10-2\sqrt{5}}}{4}.$$

b. It is straightforward to draw a line segment of length $(\sqrt{5}-1)/4$: construct a rectangle with sides 1 and 2, so the diagonal has length $\sqrt{5}$. Then subtract 1 and divide twice by 2, as shown in the figure below.

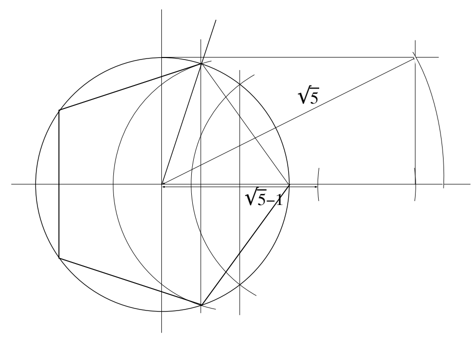

FIGURE FOR SOLUTION 0.7.15.

So if you set $\delta = \epsilon$, and $|H| \le \delta$, then equation (2) is satisfied.

c. We will show that the limit does not exist. In this case, we find

$$\begin{aligned}(A+H-A)^{-1}(A+H)^2 - A^2 &= H^{-1}(I^2 + AH + HA + H^2 - I^2)\\ &= H^{-1}(AH + HA + H^2) = A + H^{-1}AH + H^2.\end{aligned}$$

If the limit exists, it must be $2A$: choose $H = \epsilon I$ so that $H^{-1} = \epsilon^{-1}I$; then

$$A + H^{-1}AH + H^2 = 2A + \epsilon I$$

is close to $2A$.

But if you choose $H = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, you will find that

$$H^{-1}AH = \begin{bmatrix} 1/\epsilon & 0 \\ 0 & -1/\epsilon \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -A.$$

So with this H we have

$$A + H^{-1}AH + H^2 = A - A + \epsilon H$$

which is close to the zero matrix.

1.5.24

1.6.1 Let B be a set contained in a ball of radius R centered at a point $\mathbf{a}$. Then it is also contained in a ball of radius $R + |\mathbf{a}|$ centered at the origin; thus it is bounded.

1.6.2 First, remember that compact is equivalent to closed and bounded so if A is not compact then A is unbounded and/or not closed. If A is unbounded then the hint is sufficient. If A is not closed then A has a limit point $\mathbf{a}$ not in A: i.e., there exists a sequence in A that converges in $\mathbb{R}^n$ to a point $\mathbf{a} \notin A$. Use this $\mathbf{a}$ as the $\mathbf{a}$ in the hint.

1.6.3 The polynomial $p(z) = 1 + x^2y^2$ has no roots because 1 plus something positive cannot be 0. This does not contradict the fundamental theorem of algebra because although p is a polynomial in the real variables x and y, it is not a polynomial in the complex variable z: it is a polynomial in z and $\bar{z}$. It is possible to write $p(z) = 1 + x^2y^2$ in terms of z and $\bar{z}$. You can use

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i},$$

and find

$$p(z) = 1 + \frac{z^4 - 2|z|^4 + \overline{z}^4}{-16} \tag{1}$$

but you simply cannot get rid of the $\overline{z}$.

1.6.4 If $|z| \ge 4$, then

$$|p(z)| \ge |z|^5 - 4|z|^3 - 3|z| - 3 > |z|^5 - 4|z|^3 - 3|z|^3 - 3|z|^3 = |z|^3(|z|^2 - 10) \ge 6 \cdot 4^3.$$