Math 8201 **Homework 8** **PJW**

Date due: November 6, 2017. There will be a quiz on this date.

Hand in only the starred questions.

Section 4.5 8, 10, 11, 14, 15, 16, 17*, 18, 21, 24*, 30, 32, 33*, 34, 35, 41*, 42

There have been many questions on Sylow's theorem in the preliminary written exams. In recent years they have been rather straightforward. Some questions that appeared in previous years are more challenging, such as question 5 in the Spring of 1994.

5. (Spring 1994)
   (i) Let G be a finite group. Prove that the number of conjugates in G of a subgroup H equals the index of its normalizer $N_G(H)$ in G.
   (ii) Let now G be a simple group of order $1092 = 4 \cdot 3 \cdot 7 \cdot 13$.
   a) Find the number of Sylow 13-subgroups and the number of Sylow 7-subgroups of G.
   b) Prove that G has a single conjugacy class of subgroups of index 14.
   c) Prove that G has no subgroup of index 13.
   [You may assume Sylow's theorems.]

BB*. Let G be a finite group and let $O_p(G)$ denote the unique largest normal p-subgroup of G, which contains all other normal p-subgroups of G and whose existence was established in question P of sheet 4. Show that $O_p(G)$ equals the intersection of all the Sylow p-subgroups of G:

$$O_p(G) = \bigcap_{P \text{ a Sylow } p-\text{subgroup}} P.$$

[You do not have to do question P. Show that the group on the right contains every normal p-subgroup of G, and is a normal p-subgroup of G.]

CC. Let G be a finite group and H a subgroup. Let P_H be a Sylow p-subgroup of H. Prove that there exists a Sylow p-subgroup P of G such that $P_H = P \cap H$.

DD. Prove that one of the Sylow subroups of a group of order 40 is normal. Give an explicit example of a group of order 40 in which the Sylow 2-subgroup is not normal.

Section 4.6 2*, 3, 4, 5, 6 (You don't need to know the proof of Theorem 24 to do these questions.)