Math 8245 **Homework 4** **PJW**
Date due: November 26, 2018.

In the questions that follow about $SL(2,\mathbb{Z})$ we will denote

$$\alpha := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

1. Let $F(x,y)$ be a free group on generators x and y and let $\Gamma = \Gamma(F(x,y),\{x,y\})$ be the tree that is the Cayley graph with respect to these generators. For each $n \geq 0$ let $H_n = \langle x, {}^{y}x, {}^{y^2}x, \ldots, {}^{y^n}x\rangle$ as a subgroup of $F(x,y)$.
(a) Draw a picture of the quotient $H_n\backslash\Gamma$.
(b) Show that H_n is a free group of rank n.
(c) Show that $H = \langle {}^{y^i}x \mid i \in \mathbb{N}\rangle$ is a free subgroup of $F(x,y)$ of infinite rank.

2. Let F be free on generators x and y and let $\phi : F \to S_3$ be the homomorphism determined by $\phi(x) = (1,2)$ and $\phi(y) = (2,3)$. Let N be the kernel of ϕ. Find a set of free generators for N.
[You may assume without proof that N is indeed a free group and that N is generated as a *normal* subgroup of F (not as a free group) by x^2, y^2 and $xyxyxy$.]

3. Express the matrices
$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$
as products of the generators α and β of $SL(2,\mathbb{Z})$.

4. Suppose that H and J are subgroups of G. The coset graph $C(G;H,J)$ of G with respect to H and J has the left cosets xH and yJ of H and J as its vertices, and there is an edge joining xH and yJ if and only if $xH \cap yJ \neq \emptyset$. There is a left action of G on $C(G;H,J)$ as a group of graph automorphisms, given by left multiplication.
(a) Draw a picture of $C(C_{12}; C_4, C_6)$
(b) Show that there is a single G-orbit of edges in $C(G;H,J)$.
(c) Describe a fundamental domain for the action of G on $C(G;H,J)$, indicating stabilizers of all vertices and edges in your fundamental domain.
(d) Suppose that $K \lhd G$. Show that
$$K\backslash C(G;H,J) \cong C(G/K; HK/K, JK/K).$$

5. (a) Show that the coset graph $C(SL(2,\mathbb{Z}); \langle\alpha\rangle, \langle\beta\rangle)$ is a tree
(b) Let K be kernel of the homomorphism $SL(2,\mathbb{Z}) \to C_{12} = \langle x\rangle$ that sends α to x^3 and β to x^2. Find the rank of K as a free group. Find a set of matrices that are free generators for K. [Questions 4 and 5a are intended to help with this.]

6. (a) Let G be the group of permutations of $\mathbb{Z}$ generated by the two mappings α and β defined by $\alpha(x) = -x$ and $\beta(x) = -x+1$. Show that G is isomorphic to $C_2 * C_2$. Let a, b denote generators of the two copies of C_2 in this free product.
(b) Show that $G \cong C_\infty \rtimes C_2$ where C_∞ is an infinite cyclic subgroup. Identify a generator of this subgroup as a word in a and b.
(c) Show that every subgroup of $C_2 * C_2$ is isomorphic to $1, C_2, C_\infty$ or $C_2 * C_2$.

7. Is $C_2 * C_2$ isomorphic to a subgroup of $SL(2,\mathbb{Z})$? Is $C_2 * C_2$ isomorphic to a subgroup of $PSL(2,\mathbb{Z})$?
[Note (and prove if you want to) that $C_2 * C_2$ is isomorphic to the group of 2 by 2 integer matrices generated by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.]

Extra questions – do not hand in:

8. Let G be the free group on generators a and b. Prove that G is not generated by $\{a, bab\}$ or by $\{aba, bab\}$. [This shows that we cannot take the generating set $\{a, aba\}$ and replace a by either element of $\{b, bab\}$ and still have a generating set, in contrast to what happens with vector spaces. Generating sets do not form a matroid.]

9. Let $G = \langle a, b\rangle$ where $a = (1,2,3,4)$ and $b = (3,4,5,6)$. Find a Schreier transversal in terms of a and b for $\mathrm{Stab}_G(2)$, the stabilizer in G of the symbol 2.

10. Let F be a free group on a subset X. If $x \in X$ and $f \in F$, define $\sigma_x(f)$ to be the sum of the exponents of x in the reduced form of f. Prove that $f \in F'$ if and only if $\sigma_x(f) = 0$ for all x in X. [Here F' is the derived subgroup, which is generated by the commutators of elements in F.]

11. The Mathieu group M_{24} may be generated by permutations

$$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14,15,16,17,18,19,20,21)(22,23)$$

and

$$(1,2,5,7,15,20,14,23,21,11,16,19,24,6,8,4,17,3,10,13,18)(9,22,12).$$

a) Make a stabilizer chain for M_{24} and determine the lengths of the orbits $\Delta^{(i)}$.
b) What is the smallest size of a base for a group of size $|M_{24}|$ acting on 24 points?

12. The Mathieu group M_{12} may be generated by permutations

$$(1,2,3,4,5,6)(7,8,9,10,11,12) \quad \text{and} \quad (1,9,12,7,11)(6,2,8,3,5).$$

Same question as for M_{24}.