10.4 TENSOR PRODUCTS OF MODULES

In this section we study the tensor product of two modules M and N over a ring (not necessarily commutative) containing 1. Formation of the tensor product is a general construction that, loosely speaking, enables one to form another module in which one can take "products" mn of elements $m \in M$ and $n \in N$. The general construction involves various left- and right- module actions, and it is instructive, by way of motivation, to first consider an important special case: the question of "extending scalars" or "changing the base."

Suppose that the ring R is a subring of the ring S. Throughout this section, we always assume that $1_R = 1_S$ (this ensures that S is a unital R-module).

If N is a left S-module, then N can also be naturally considered as a left R-module since the elements of R (being elements of S) act on N by assumption. The S-module axioms for N include the relations

$$(s_1 + s_2)n = s_1 n + s_2 n \quad \text{and} \quad s(n_1 + n_2) = sn_1 + sn_2 \tag{10.1}$$

for all $s, s_1, s_2 \in S$ and all $n, n_1, n_2 \in N$, and the relation

$$(s_1 s_2)n = s_1(s_2 n) \quad \text{for all } s_1, s_2 \in S, \text{ and all } n \in N. \tag{10.2}$$

A particular case of the latter relation is

$$(sr)n = s(rn) \quad \text{for all } s \in S, r \in R \text{ and } n \in N. \tag{10.2$'$}$$

More generally, if $f : R \to S$ is a ring homomorphism from R into S with $f(1_R) = 1_S$ (for example the injection map if R is a subring of S as above) then it is easy to see that N can be considered as an R-module with $rn = f(r)n$ for $r \in R$ and $n \in N$. In this situation S can be considered as an *extension* of the ring R and the resulting R-module is said to be obtained from N by *restriction of scalars* from S to R.

Suppose now that R is a subring of S and we try to reverse this, namely we start with an R-module N and attempt to define an S-module structure on N that extends the action of R on N to an action of S on N (hence "extending the scalars" from R to S). In general this is impossible, even in the simplest situation: the ring R itself is an R-module but is usually not an S-module for the larger ring S. For example, $\mathbb{Z}$ is a $\mathbb{Z}$-module but it cannot be made into a $\mathbb{Q}$-module (if it could, then $\frac{1}{2} \circ 1 = z$ would be an element of $\mathbb{Z}$ with $z + z = 1$, which is impossible). Although $\mathbb{Z}$ itself cannot be made into a $\mathbb{Q}$-module it is *contained* in a $\mathbb{Q}$-module, namely $\mathbb{Q}$ itself. Put another way, there is an injection (also called an *embedding*) of the $\mathbb{Z}$-module $\mathbb{Z}$ into the $\mathbb{Q}$-module $\mathbb{Q}$ (and similarly the ring R can always be embedded as an R-submodule of the S-module S). This raises the question of whether an arbitrary R-module N can be embedded as an R-submodule of some S-module, or more generally, the question of what R-module homomorphisms exist from N to S-modules. For example, suppose N is a nontrivial *finite* abelian group, say $N = \mathbb{Z}/2\mathbb{Z}$, and consider possible $\mathbb{Z}$-module homomorphisms (i.e., abelian group homomorphisms) of N into some $\mathbb{Q}$-module. A $\mathbb{Q}$-module is just a vector space over $\mathbb{Q}$ and every nonzero element in a vector space over $\mathbb{Q}$ has infinite (additive) order. Since every element of N has finite order, every element of N must map to 0 under such a homomorphism. In other words there are *no* nonzero $\mathbb{Z}$-module homomorphisms from this N to *any* $\mathbb{Q}$-module, much less embeddings of N identifying

N as a submodule of a $\mathbb{Q}$-module. The two $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ exhibit extremely different behaviors when we try to "extend scalars" from $\mathbb{Z}$ to $\mathbb{Q}$: the first module maps injectively into some $\mathbb{Q}$-module, the second always maps to 0 in a $\mathbb{Q}$-module.

We now construct for a general R-module N an S-module that is the "best possible" target in which to try to embed N. We shall also see that this module determines *all* of the possible R-module homomorphisms of N into S-modules, in particular determining when N is contained in some S-module (cf. Corollary 9). In the case of $R = \mathbb{Z}$ and $S = \mathbb{Q}$ this construction will give us $\mathbb{Q}$ when applied to the module $N = \mathbb{Z}$, and will give us 0 when applied to the module $N = \mathbb{Z}/2\mathbb{Z}$ (Examples 2 and 3 following Corollary 9).

If the R-module N were already an S-module then of course there is no difficulty in "extending" the scalars from R to S, so we begin the construction by returning to the basic module axioms in order to examine whether we can define "products" of the form sn, for $s \in S$ and $n \in N$. These axioms start with an abelian group N together with a map from $S \times N$ to N, where the image of the pair (s, n) is denoted by sn. It is therefore natural to consider the free $\mathbb{Z}$-module (i.e., , the free abelian group) on the set $S \times N$, i.e., the collection of all finite commuting sums of elements of the form (s_i, n_i) where $s_i \in S$ and $n_i \in N$. This is an abelian group where there are no relations between any distinct pairs (s, n) and (s', n'), i.e., no relations between the "formal products" sn, and in this abelian group the original module N has been thoroughly distinguished from the new "coefficients" from S. To satisfy the relations necessary for an S-module structure imposed in equation (1) and the compatibility relation with the action of R on N in (2′), we must take the quotient of this abelian group by the subgroup H generated by all elements of the form

$$\begin{gathered}(s_1 + s_2, n) - (s_1, n) - (s_2, n),\\ (s, n_1 + n_2) - (s, n_1) - (s, n_2), \text{ and}\\ (sr, n) - (s, rn),\end{gathered} \tag{10.3}$$

for $s, s_1, s_2 \in S$, $n, n_1, n_2 \in N$ and $r \in R$, where rn in the last element refers to the R-module structure already defined on N.

The resulting quotient group is denoted by $S \otimes_R N$ (or just $S \otimes N$ if R is clear from the context) and is called the *tensor product of S and N over R*. If $s \otimes n$ denotes the coset containing (s, n) in $S \otimes_R N$ then by definition of the quotient we have forced the relations

$$\begin{gathered}(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n,\\ s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2, \text{ and}\\ sr \otimes n = s \otimes rn.\end{gathered} \tag{10.4}$$

The elements of $S \otimes_R N$ are called *tensors* and can be written (non-uniquely in general) as finite sums of "simple tensors" of the form $s \otimes n$ with $s \in S$, $n \in N$.

We now show that the tensor product $S \otimes_R N$ is naturally a left S-module under the action defined by

$$s\Big(\sum_{\text{finite}} s_i \otimes n_i\Big) = \sum_{\text{finite}} (ss_i) \otimes n_i. \tag{10.5}$$

We first check this is well defined, i.e., independent of the representation of the element of $S \otimes_R N$ as a sum of simple tensors. Note first that if s' is any element of S then

$$(s'(s_1 + s_2), n) - (s's_1, n) - (s's_2, n) \ \big(= (s's_1 + s's_2, n) - (s's_1, n) - (s's_2, n)\big),$$
$$(s's, n_1 + n_2) - (s's, n_1) - (s's, n_2), \text{ and}$$
$$(s'(sr), n) - (s's, rn) \ \big(= ((s's)r, n) - (s's, rn)\big)$$

each belongs to the set of generators in (3), so in particular each lies in the subgroup H. This shows that multiplying the first entries of the generators in (3) on the left by s' gives another element of H (in fact another generator). Since any element of H is a sum of elements as in (3), it follows that for any element $\sum(s_i, n_i)$ in H also $\sum(s's_i, n_i)$ lies in H. Suppose now that $\sum s_i \otimes n_i = \sum s_i' \otimes n_i'$ are two representations for the same element in $S \otimes_R N$. Then $\sum(s_i, n_i) - \sum(s_i', n_i')$ is an element of H, and by what we have just seen, for any $s \in S$ also $\sum(ss_i, n_i) - \sum(ss_i', n_i')$ is an element of H. But this means that $\sum ss_i \otimes n_i = \sum ss_i' \otimes n_i'$ in $S \otimes_R N$, so the expression in (5) is indeed well defined.

It is now straightforward using the relations in (4) to check that the action defined in (5) makes $S \otimes_R N$ into a left S-module. For example, on the simple tensor $s_i \otimes n_i$,

$$\begin{aligned}
(s + s')\,(s_i \otimes n_i) &= ((s + s')s_i) \otimes n_i && \text{by definition (5)}\\
&= (ss_i + s's_i) \otimes n_i \\
&= ss_i \otimes n_i + s's_i \otimes n_i && \text{by the first relation in (4)}\\
&= s\,(s_i \otimes n_i) + s'\,(s_i \otimes n_i) && \text{by definition (5)}.
\end{aligned}$$

The module $S \otimes_R N$ is called *the (left) S-module obtained by extension of scalars from the (left) R-module N.*

There is a natural map $\iota : N \to S \otimes_R N$ defined by $n \mapsto 1 \otimes n$ (i.e., first map $n \in N$ to the element $(1, n)$ in the free abelian group and then pass to the quotient group). Since $1 \otimes rn = r \otimes n = r(1 \otimes n)$ by (4) and (5), it is easy to check that ι is an R-module homomorphism from N to $S \otimes_R N$. Since we have passed to a quotient group, however, ι is not injective in general. Hence, while there is a natural R-module homomorphism from the original left R-module N to the left S-module $S \otimes_R N$, in general $S \otimes_R N$ need not contain (an isomorphic copy of) N. On the other hand, the relations in equation (3) were the *minimal* relations that we had to impose in order to obtain an S-module, so it is reasonable to expect that the tensor product $S \otimes_R N$ is the "best possible" S-module to serve as target for an R-module homomorphism from N. The next theorem makes this more precise by showing that any other R-module homomorphism from N factors through this one, and is referred to as the *universal property* for the tensor product $S \otimes_R N$. The analogous result for the general tensor product is given in Theorem 10.

Theorem 8. Let R be a subring of S, let N be a left R-module and let $\iota : N \to S \otimes_R N$ be the R-module homomorphism defined by $\iota(n) = 1 \otimes n$. Suppose that L is any left S-module (hence also an R-module) and that $\varphi : N \to L$ is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism $\Phi : S \otimes_R N \to L$ such that φ factors through Φ, i.e., $\varphi = \Phi \circ \iota$ and the diagram

$$\begin{array}{ccc} N & \xrightarrow{\ \iota\ } & S \otimes_R N \\ & \searrow^{\varphi} & \downarrow \Phi \\ & & L \end{array}$$

commutes. Conversely, if $\Phi : S \otimes_R N \to L$ is an S-module homomorphism then $\varphi = \Phi \circ \iota$ is an R-module homomorphism from N to L.

Proof: Suppose $\varphi : N \to L$ is an R-module homomorphism to the S-module L. By the universal property of free modules (Theorem 6 in Section 3) there is a $\mathbb{Z}$-module homomorphism from the free $\mathbb{Z}$-module F on the set $S \times N$ to L that sends each generator (s, n) to $s\varphi(n)$. Since φ is an R-module homomorphism, the generators of the subgroup H in equation (3) all map to zero in L. Hence this $\mathbb{Z}$-module homomorphism factors through H, i.e., there is a well defined $\mathbb{Z}$-module homomorphism Φ from $F/H = S \otimes_R N$ to L satisfying $\Phi(s \otimes n) = s\varphi(n)$. Moreover, on simple tensors we have

$$s'\Phi(s \otimes n) = s'(s\varphi(n)) = (s's)\varphi(n) = \Phi((s's) \otimes n) = \Phi(s'(s \otimes n)).$$

for any $s' \in S$. Since Φ is additive it follows that Φ is an S-module homomorphism, which proves the existence statement of the theorem. The module $S \otimes_R N$ is generated as an S-module by elements of the form $1 \otimes n$, so any S-module homomorphism is uniquely determined by its values on these elements. Since $\Phi(1 \otimes n) = \varphi(n)$, it follows that the S-module homomorphism Φ is uniquely determined by φ, which proves the uniqueness statement of the theorem. The converse statement is immediate.

The universal property of $S \otimes_R N$ in Theorem 8 shows that R-module homomorphisms of N into S-modules arise from S-module homomorphisms from $S \otimes_R N$. In particular this determines when it is possible to map N injectively into some S-module:

Corollary 9. Let $\iota : N \to S \otimes_R N$ be the R-module homomorphism in Theorem 8. Then $N/\ker \iota$ is the unique largest quotient of N that can be embedded in any S-module. In particular, N can be embedded as an R-submodule of some left S-module if and only if ι is injective (in which case N is isomorphic to the R-submodule $\iota(N)$ of the S-module $S \otimes_R N$).

Proof: The quotient $N/\ker \iota$ is mapped injectively (by ι) into the S-module $S \otimes_R N$. Suppose now that φ is an R-module homomorphism injecting the quotient $N/\ker \varphi$ of N into an S-module L. Then, by Theorem 8, $\ker \iota$ is mapped to 0 by φ, i.e., $\ker \iota \subseteq \ker \varphi$. Hence $N/\ker \varphi$ is a quotient of $N/\ker \iota$ (namely, the quotient by the submodule $\ker \varphi / \ker \iota$). It follows that $N/\ker \iota$ is the unique largest quotient of N that can be embedded in any S-module. The last statement in the corollary follows immediately.

Examples

(1) For any ring R and any left R-module N we have $R \otimes_R N \cong N$ (so "extending scalars from R to R" does not change the module). This follows by taking φ to be the identity map from N to itself (and $S = R$) in Theorem 8: ι is then an isomorphism with inverse isomorphism given by Φ. In particular, if A is any abelian group (i.e., a $\mathbb{Z}$-module), then $\mathbb{Z} \otimes_{\mathbb{Z}} A = A$.

(2) Let $R = \mathbb{Z}$, $S = \mathbb{Q}$ and let A be a finite abelian group of order n. In this case the $\mathbb{Q}$-module $\mathbb{Q} \otimes_{\mathbb{Z}} A$ obtained by extension of scalars from the $\mathbb{Z}$-module A is 0. To see this, observe first that in any tensor product $1 \otimes 0 = 1 \otimes (0+0) = 1 \otimes 0 + 1 \otimes 0$, by the second relation in (4), so

$$1 \otimes 0 = 0.$$

Now, for any simple tensor $q \otimes a$ we can write the rational number q as $(q/n)n$. Then since $na = 0$ in A by Lagrange's Theorem, we have

$$q \otimes a = (\frac{q}{n} \cdot n) \otimes a = \frac{q}{n} \otimes (na) = (q/n) \otimes 0 = (q/n)(1 \otimes 0) = 0.$$

It follows that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$. In particular, the map $\iota : A \to S \otimes_R A$ is the zero map. By Theorem 8, we see again that any homomorphism of a finite abelian group into a rational vector space is the zero map. In particular, if A is nontrivial, then the original $\mathbb{Z}$-module A is not contained in the $\mathbb{Q}$-module obtained by extension of scalars.

(3) *Extension of scalars for free modules:* If $N \cong R^n$ is a free module of rank n over R then $S \otimes_R N \cong S^n$ is a free module of rank n over S. We shall prove this shortly (Corollary 18) when we discuss tensor products of direct sums. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$. In this case the module obtained by extension of scalars contains (an isomorphic copy of) the original R-module N. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ and $\mathbb{Z}^n$ is a subgroup of the abelian group $\mathbb{Q}^n$.

(4) *Extension of scalars for vector spaces:* As a special case of the previous example, let F be a subfield of the field K and let V be an n-dimensional vector space over F (i.e., $V \cong F^n$). Then $K \otimes_F V \cong K^n$ is a vector space over the larger field K of the same dimension, and the original vector space V is contained in $K \otimes_F V$ as an F-vector subspace.

(5) *Induced modules for finite groups:* Let R be a commutative ring with 1, let G be a finite group and let H be a subgroup of G. As in Section 7.2 we may form the group ring RG and its subring RH. For any RH-module N define the *induced module* $RG \otimes_{RH} N$. In this way we obtain an RG-module for each RH-module N. We shall study properties of induced modules and some of their important applications to group theory in Chapters 17 and 19.

The general tensor product construction follows along the same lines as the extension of scalars above, but before describing it we make two observations from this special case. The first is that the construction of $S \otimes_R N$ as an *abelian group* involved only the elements in equation (3), which in turn only required S to be a *right* R-module and N to be a *left* R-module. In a similar way we shall construct an *abelian group* $M \otimes_R N$ for *any right* R-module M and *any left* R-module N. The second observation is that the S-*module* structure on $S \otimes_R N$ defined by equation (5) required only a *left* S-module structure on S together with a "compatibility relation"

$$s'(sr) = (s's)r \qquad \text{for } s, s' \in S, r \in R,$$

between this left S-module structure and the right R-module structure on S (this was needed in order to deduce that (5) was well defined). We first consider the general construction of $M \otimes_R N$ as an abelian group, after which we shall return to the question of when this abelian group can be given a module structure.

Suppose then that N is a left R-module and that M is a right R-module. The quotient of the free $\mathbb{Z}$-module on the set $M \times N$ by the subgroup generated by all elements of the form

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n),\\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2), \text{ and} \\
&(mr, n) - (m, rn),
\end{aligned} \tag{10.6}$$

for $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $r \in R$ is an abelian group, denoted by $M \otimes_R N$, or simply $M \otimes N$ if the ring R is clear from the context, and is called the *tensor product of M and N over R*. The elements of $M \otimes_R N$ are called *tensors*, and the coset, $m \otimes n$, of (m, n) in $M \otimes_R N$ is called a simple tensor. We have the relations

$$\begin{aligned}
(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n,\\
m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2, \text{ and} \\
mr \otimes n &= m \otimes rn.
\end{aligned} \tag{10.7}$$

Every tensor can be written (non-uniquely in general) as a finite sum of simple tensors.

Remark: We emphasize that care must be taken when working with tensors, since each $m \otimes n$ represents a *coset* in some quotient group, and so we may have $m \otimes n = m' \otimes n'$ where $m \neq m'$ or $n \neq n'$. More generally, an element of $M \otimes N$ may be expressible in many different ways as a sum of simple tensors. In particular, care must be taken when defining maps from $M \otimes_R N$ to another group or module, since a map from $M \otimes N$ which is described on the generators $m \otimes n$ in terms of m and n is not well defined unless it is shown to be independent of the particular choice of $m \otimes n$ as a coset representative.

Another point where care must be exercised is in reference to the element $m \otimes n$ when the modules M and N or the ring R are not clear from the context. The first two examples of extension of scalars give an instance where M is a submodule of a larger module M', and for some $m \in M$ and $n \in N$ we have $m \otimes n = 0$ in $M' \otimes_R N$ but $m \otimes n$ is *nonzero* in $M \otimes_R N$. This is possible because the symbol "$m \otimes n$" represents different cosets, hence possibly different elements, in the two tensor products. In particular, these two examples show that $M \otimes_R N$ need not be a subgroup of $M' \otimes_R N$ even when M is a submodule of M' (cf. also Exercise 2).

Mapping $M \times N$ to the free $\mathbb{Z}$-module on $M \times N$ and then passing to the quotient defines a map $\iota : M \times N \to M \otimes_R N$ with $\iota(m, n) = m \otimes n$. This map is in general not a group homomorphism, but it is additive in both m and n separately and satisfies $\iota(mr, n) = mr \otimes n = m \otimes rn = \iota(m, rn)$. Such maps are given a name:

Definition. Let M be a right R-module, let N be a left R-module and let L be an abelian group (written additively). A map $\varphi : M \times N \to L$ is called *R-balanced* or *middle linear with respect to R* if

$$\begin{aligned}\varphi(m_1 + m_2, n) &= \varphi(m_1, n) + \varphi(m_2, n)\\ \varphi(m, n_1 + n_2) &= \varphi(m, n_1) + \varphi(m, n_2)\\ \varphi(m, rn) &= \varphi(mr, n)\end{aligned}$$

for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N$, and $r \in R$.

With this terminology, it follows immediately from the relations in (7) that the map $\iota : M \times N \to M \otimes_R N$ is R-balanced. The next theorem proves the extremely useful *universal property of the tensor product* with respect to balanced maps.

Theorem 10. Suppose R is a ring with 1, M is a right R-module, and N is a left R-module. Let $M \otimes_R N$ be the tensor product of M and N over R and let $\iota : M \times N \to M \otimes_R N$ be the R-balanced map defined above.

(1) If $\Phi : M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L then the composite map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N$ to L.

(2) Conversely, suppose L is an abelian group and $\varphi : M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that φ factors through ι, i.e., $\varphi = \Phi \circ \iota$ as in (1).

Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ in the commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\ \iota\ } & M \otimes_R N \\ & \searrow^{\varphi} & \downarrow{\Phi} \\ & & L \end{array}$$

establishes a bijection

$$\left\{\begin{matrix} R\text{-balanced maps} \\ \varphi : M \times N \to L \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{group homomorphisms} \\ \Phi : M \otimes_R N \to L \end{matrix}\right\}.$$

Proof: The proof of (1) is immediate from the properties of ι above. For (2), the map φ defines a unique $\mathbb{Z}$-module homomorphism $\tilde{\varphi}$ from the free group on $M \times N$ to L (Theorem 6 in Section 3) such that $\tilde{\varphi}(m, n) = \varphi(m, n) \in L$. Since φ is R-balanced, $\tilde{\varphi}$ maps each of the elements in equation (6) to 0; for example

$$\tilde{\varphi}\left((mr, n) - (m, rn)\right) = \varphi(mr, n) - \varphi(m, rn) = 0.$$

It follows that the kernel of $\tilde{\varphi}$ contains the subgroup generated by these elements, hence $\tilde{\varphi}$ induces a homomorphism Φ on the quotient group $M \otimes_R N$ to L. By definition we then have

$$\Phi(m \otimes n) = \tilde{\varphi}(m, n) = \varphi(m, n),$$

i.e., $\varphi = \Phi \circ \iota$. The homomorphism Φ is uniquely determined by this equation since the elements $m \otimes n$ generate $M \otimes_R N$ as an abelian group. This completes the proof.

Theorem 10 is extremely useful in defining homomorphisms on $M \otimes_R N$ since it replaces the often tedious check that maps defined on simple tensors $m \otimes n$ are well defined with a check that a related map defined on ordered pairs (m, n) is balanced.

The first consequence of the universal property in Theorem 10 is a characterization of the tensor product $M \otimes_R N$ as an abelian group:

Corollary 11. Suppose D is an abelian group and $\iota' : M \times N \to D$ is an R-balanced map such that

(i) the image of ι' generates D as an abelian group, and
(ii) every R-balanced map defined on $M \times N$ factors through ι' as in Theorem 10.

Then there is an isomorphism $f : M \otimes_R N \cong D$ of abelian groups with $\iota' = f \circ \iota$.

Proof: Since $\iota' : M \times N \to D$ is a balanced map, the universal property in (2) of Theorem 10 implies there is a (unique) homomorphism $f : M \otimes_R N \to D$ with $\iota' = f \circ \iota$. In particular $\iota'(m, n) = f(m \otimes n)$ for every $m \in M$, $n \in N$. By the first assumption on ι', these elements generate D as an abelian group, so f is a surjective map. Now, the balanced map $\iota : M \times N \to M \otimes_R N$ together with the second assumption on ι' implies there is a (unique) homomorphism $g : D \to M \otimes_R N$ with $\iota = g \circ \iota'$. Then $m \otimes n = (g \circ f)(m \otimes n)$. Since the simple tensors $m \otimes n$ generate $M \otimes_R N$, it follows that $g \circ f$ is the identity map on $M \otimes_R N$ and so f is injective, hence an isomorphism. This establishes the corollary.

We now return to the question of giving the abelian group $M \otimes_R N$ a *module* structure. As we observed in the special case of extending scalars from R to S for the R-module N, the S-module structure on $S \otimes_R N$ required only a left S-module structure on S together with the compatibility relation $s'(sr) = (s's)r$ for $s, s' \in S$ and $r \in R$. In this special case this relation was simply a consequence of the associative law in the ring S. To obtain an S-module structure on $M \otimes_R N$ more generally we impose a similar structure on M:

Definition. Let R and S be any rings with 1. An abelian group M is called an (S, R)-*bimodule* if M is a left S-module, a right R-module, and $s(mr) = (sm)r$ for all $s \in S$, $r \in R$ and $m \in M$.

Examples

(1) Any ring S is an (S, R)-bimodule for any subring R with $1_R = 1_S$ by the associativity of the multiplication in S. More generally, if $f : R \to S$ is any ring homomorphism with $f(1_R) = 1_S$ then S can be considered as a right R-module with the action $s \cdot r = sf(r)$, and with respect to this action S becomes an (S, R)-bimodule.
(2) Let I be an ideal (two-sided) in the ring R. Then the quotient ring R/I is an $(R/I, R)$-bimodule. This is easy to see directly and is also a special case of the previous example (with respect to the canonical projection homomorphism $R \to R/I$).
(3) Suppose that R is a commutative ring. Then a left (respectively, right) R-module M can always be given the structure of a right (respectively, left) R-module by defining $mr = rm$ (respectively, $rm = mr$), for all $m \in M$ and $r \in R$, and this makes M into

an (R, R)-bimodule. Hence every module (right or left) over a commutative ring R has at least one natural (R, R)-bimodule structure.

(4) Suppose that M is a left S-module and R is a subring contained in the *center* of S (for example, if S is commutative). Then in particular R is commutative so M can be given a right R-module structure as in the previous example. Then for any $s \in S$, $r \in R$ and $m \in M$ by definition of the right action of R we have

$$(sm)r = r(sm) = (rs)m = (sr)m = s(rm) = s(mr)$$

(note that we have used the fact that r commutes with s in the middle equality). Hence M is an (S, R)-bimodule with respect to this definition of the right action of R.

Since the situation in Example 3 occurs so frequently, we give this bimodule structure a name:

Definition. Suppose M is a left (or right) R-module over the commutative ring R. Then the (R, R)-bimodule structure on M defined by letting the left and right R-actions coincide, i.e., $mr = rm$ for all $m \in M$ and $r \in R$, will be called the *standard* R-module structure on M.

Suppose now that N is a left R-module and M is an (S, R)-bimodule. Then just as in the example of extension of scalars the (S, R)-bimodule structure on M implies that

$$s\left(\sum_{\text{finite}} m_i \otimes n_i\right) = \sum_{\text{finite}} (sm_i) \otimes n_i \qquad (10.8)$$

gives a well defined action of S under which *$M \otimes_R N$ is a left S-module*. Note that Theorem 10 may be used to give an alternate proof that (8) is well defined, replacing the direct calculations on the relations defining the tensor product with the easier check that a map is R-balanced, as follows. It is very easy to see that for each fixed $s \in S$ the map $(m, n) \mapsto sm \otimes n$ is an R-balanced map from $M \times N$ to $M \otimes_R N$. By Theorem 10 there is a well defined group homomorphism λ_s from $M \otimes_R N$ to itself such that $\lambda_s(m \otimes n) = sm \otimes n$. Since the right side of (8) is then $\lambda_s(\sum m_i \otimes n_i)$, the fact that λ_s is well defined shows that this expression is indeed independent of the representation of the tensor $\sum m_i \otimes n_i$ as a sum of simple tensors. Because λ_s is additive, equation (8) holds.

By a completely parallel argument, if M is a right R-module and N is an (R, S)-bimodule then the tensor product $M \otimes_R N$ has the structure of a *right* S-module, where $(\sum m_i \otimes n_i)s = \sum m_i \otimes (n_i s)$.

Before giving some more examples of tensor products it is worthwhile to highlight one frequently encountered special case of the previous discussion, namely the case when M and N are two left modules over a *commutative* ring R and $S = R$ (in some works on tensor products this is the only case considered). Then the standard R-module structure on M defined previously gives M the structure of an (R, R)-bimodule, so in this case the tensor product $M \otimes_R N$ always has the structure of a left R-module.

The corresponding map $\iota : M \times N \to M \otimes_R N$ maps $M \times N$ into an R-module and is additive in each factor. Since $r(m \otimes n) = rm \otimes n = mr \otimes n = m \otimes rn$ it also satisfies

$$r\iota(m, n) = \iota(rm, n) = \iota(m, rn).$$

Such maps are given a name:

Definition. Let R be a commutative ring with 1 and let M, N, and L be left R-modules. The map $\varphi : M \times N \to L$ is called *R-bilinear* if it is R-linear in each factor, i.e., if

$$\varphi(r_1m_1 + r_2m_2, n) = r_1\varphi(m_1, n) + r_2\varphi(m_2, n), \quad \text{and}$$
$$\varphi(m, r_1n_1 + r_2n_2) = r_1\varphi(m, n_1) + r_2\varphi(m, n_2)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r_1, r_2 \in R$.

With this terminology Theorem 10 gives

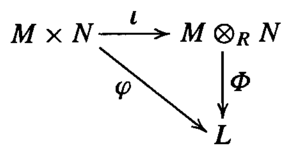
Corollary 12. Suppose R is a commutative ring. Let M and N be two left R-modules and let $M \otimes_R N$ be the tensor product of M and N over R, where M is given the standard R-module structure. Then $M \otimes_R N$ is a left R-module with

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn),$$

and the map $\iota : M \times N \to M \otimes_R N$ with $\iota(m, n) = m \otimes n$ is an R-bilinear map. If L is any left R-module then there is a bijection

$$\left\{ \begin{matrix} R\text{-bilinear maps} \\ \varphi : M \times N \to L \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} R\text{-module homomorphisms} \\ \Phi : M \otimes_R N \to L \end{matrix} \right\}$$

where the correspondence between φ and Φ is given by the commutative diagram



Proof: We have shown $M \otimes_R N$ is an R-module and that ι is bilinear. It remains only to check that in the bijective correspondence in Theorem 10 the bilinear maps correspond with the R-module homomorphisms. If $\varphi : M \times N \to L$ is bilinear then it is an R-balanced map, so the corresponding $\Phi : M \otimes_R N$ is a group homomorphism. Moreover, on simple tensors $\Phi((rm) \otimes n) = \varphi(rm, n) = r\varphi(m, n) = r\Phi(m \otimes n)$, where the middle equality holds because φ is R-linear in the first variable. Since Φ is additive this extends to sums of simple tensors to show Φ is an R-module homomorphism. Conversely, if Φ is an R-module homomorphism it is an exercise to see that the corresponding balanced map φ is bilinear.

Examples

(1) In any tensor product $M \otimes_R N$ we have $m \otimes 0 = m \otimes (0 + 0) = (m \otimes 0) + (m \otimes 0)$, so $m \otimes 0 = 0$. Likewise $0 \otimes n = 0$.

(2) We have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, since $3a = a$ for $a \in \mathbb{Z}/2\mathbb{Z}$ so that

$$a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes 0 = 0$$

and every simple tensor is reduced to 0. In particular $1 \otimes 1 = 0$. It follows that there are no nonzero balanced (or bilinear) maps from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ to any abelian group.

On the other hand, consider the tensor product $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, which is generated as an abelian group by the elements $0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0$ and $1 \otimes 1$. In this case $1 \otimes 1 \neq 0$ since, for example, the map $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $(a, b) \mapsto ab$ is clearly nonzero and linear in both a and b. Since $2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0$, the element $1 \otimes 1$ is of order 2. Hence $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

(3) In general,

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z},$$

where d is the g.c.d. of the integers m and n. To see this, observe first that

$$a \otimes b = a \otimes (b \cdot 1) = (ab) \otimes 1 = ab(1 \otimes 1),$$

from which it follows that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is a cyclic group with $1 \otimes 1$ as generator. Since $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$ and similarly $n(1 \otimes 1) = 1 \otimes n = 0$, we have $d(1 \otimes 1) = 0$, so the cyclic group has order dividing d. The map $\varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ defined by $\varphi(a \bmod m, b \bmod n) = ab \bmod d$ is well defined since d divides both m and n. It is clearly $\mathbb{Z}$-bilinear. The induced map $\Phi : \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ from Corollary 12 maps $1 \otimes 1$ to the element $1 \in \mathbb{Z}/d\mathbb{Z}$, which is an element of order d. In particular $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ has order at least d. Hence $1 \otimes 1$ is an element of order d and Φ gives an isomorphism $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

(4) In $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ a simple tensor has the form $(a/b \bmod \mathbb{Z}) \otimes (c/d \bmod \mathbb{Z})$ for some rational numbers a/b and c/d. Then

$$\begin{aligned}(\frac{a}{b} \bmod \mathbb{Z}) \otimes (\frac{c}{d} \bmod \mathbb{Z}) &= d(\frac{a}{bd} \bmod \mathbb{Z}) \otimes (\frac{c}{d} \bmod \mathbb{Z}) \\ &= (\frac{a}{bd} \bmod \mathbb{Z}) \otimes d(\frac{c}{d} \bmod \mathbb{Z}) = (\frac{a}{bd} \bmod \mathbb{Z}) \otimes 0 = 0\end{aligned}$$

and so

$$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

In a similar way, $A \otimes_{\mathbb{Z}} B = 0$ for any *divisible* abelian group A and *torsion* abelian group B (an abelian group in which every element has finite order). For example

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

(5) The structure of a tensor product can vary considerably depending on the ring over which the tensors are taken. For example $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic as left $\mathbb{Q}$-modules (both are one dimensional vector spaces over $\mathbb{Q}$) — cf. the exercises. On the other hand we shall see at the end of this section that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ are not isomorphic $\mathbb{C}$-modules (the former is a 1-dimensional vector space over $\mathbb{C}$ and the latter is 2-dimensional over $\mathbb{C}$).

(6) *General extension of scalars or change of base:* Let $f : R \to S$ be a ring homomorphism with $f(1_R) = 1_S$. Then $s \cdot r = sf(r)$ gives S the structure of a right R-module with respect to which S is an (S, R)-bimodule. Then for any left R-module N, the resulting tensor product $S \otimes_R N$ is a left S-module obtained by *changing the base* from R to S. This gives a slight generalization of the notion of extension of scalars (where R was a subring of S).

(7) Let $f : R \to S$ be a ring homomorphism as in the preceding example. Then we have $S \otimes_R R \cong S$ as left S-modules, as follows. The map $\varphi : S \times R \to S$ defined by $(s, r) \mapsto sr$ (where $sr = sf(r)$ by definition of the right R-action on S), is an R-balanced map, as is easily checked. For example,

$$\varphi(s_1 + s_2, r) = (s_1 + s_2)r = s_1 r + s_2 r = \varphi(s_1, r) + \varphi(s_2, r)$$

and

$$\varphi(sr, r') = (sr)r' = s(rr') = \varphi(s, rr').$$

By Theorem 10 we have an associated group homomorphism $\Phi : S \otimes_R R \to S$ with $\Phi(s \otimes r) = sr$. Since $\Phi(s'(s \otimes r)) = \Phi(s's \otimes r) = s'sr = s'\Phi(s \otimes r)$, it follows that Φ is also an S-module homomorphism. The map $\Phi' : S \to S \otimes_R R$ with $s \mapsto s \otimes 1$ is an S-module homomorphism that is inverse to Φ because $\Phi \circ \Phi'(s) = \Phi(s \otimes 1) = s$ gives $\Phi\Phi' = 1$, and

$$\Phi' \circ \Phi(s \otimes r) = \Phi'(sr) = sr \otimes 1 = s \otimes r$$

shows that $\Phi'\Phi$ is the identity on simple tensors, hence $\Phi'\Phi = 1$.

(8) Let R be a ring (not necessarily commutative), let I be a two sided ideal in R, and let N be a left R-module. Then as previously mentioned, R/I is an $(R/I, R)$-bimodule, so the tensor product $R/I \otimes_R N$ is a left R/I-module. This is an example of "extension of scalars" with respect to the natural projection homomorphism $R \to R/I$.

Define

$$IN = \left\{ \sum_{\text{finite}} a_i n_i \mid a_i \in I, n_i \in N \right\},$$

which is easily seen to be a left R-submodule of N (cf. Exercise 5, Section 1). Then

$$(R/I) \otimes_R N \cong N/IN,$$

as left R-modules, as follows. The tensor product is generated as an abelian group by the simple tensors $(r \bmod I) \otimes n = r(1 \otimes n)$ for $r \in R$ and $n \in N$ (viewing the R/I-module tensor product as an R-module on which I acts trivially). Hence the elements $1 \otimes n$ generate $(R/I) \otimes_R N$ as an R/I-module. The map $N \to (R/I) \otimes_R N$ defined by $n \mapsto 1 \otimes n$ is a left R-module homomorphism and, by the previous observation, is surjective. Under this map $a_i n_i$ with $a_i \in I$ and $n_i \in N$ maps to $1 \otimes a_i n_i = a_i \otimes n_i = 0$, and so IN is contained in the kernel. This induces a surjective R-module homomorphism $f : N/IN \to (R/I) \otimes_R N$ with $f(n \bmod I) = 1 \otimes n$. We show f is an isomorphism by exhibiting its inverse. The map $(R/I) \times N \to N/IN$ defined by mapping $(r \bmod I, n)$ to $(rn \bmod IN)$ is well defined and easily checked to be R-balanced. It follows by Theorem 10 that there is an associated group homomorphism $g : (R/I) \otimes N \to N/IN$ with $g((r \bmod I) \otimes n) = rn \bmod IN$. As usual, $fg = 1$ and $gf = 1$, so f is a bijection and $(R/I) \otimes_R N \cong N/IN$, as claimed.

As an example, let $R = \mathbb{Z}$ with ideal $I = m\mathbb{Z}$ and let N be the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. Then $IN = m(\mathbb{Z}/n\mathbb{Z}) = (m\mathbb{Z} + n\mathbb{Z})/n\mathbb{Z} = d\mathbb{Z}/n\mathbb{Z}$ where d is the g.c.d. of m and n. Then $N/IN \cong \mathbb{Z}/d\mathbb{Z}$ and we recover the isomorphism $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ of Example 3 above.

We now establish some of the basic properties of tensor products. Note the frequent application of Theorem 10 to establish the existence of homomorphisms.

Theorem 13. *(The "Tensor Product" of Two Homomorphisms)* Let M, M' be right R-modules, let N, N' be left R-modules, and suppose $\varphi : M \to M'$ and $\psi : N \to N'$ are R-module homomorphisms.

(1) There is a unique group homomorphism, denoted by $\varphi \otimes \psi$, mapping $M \otimes_R N$ into $M' \otimes_R N'$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$ for all $m \in M$ and $n \in N$.

(2) If M, M' are also (S, R)-bimodules for some ring S and φ is also an S-module homomorphism, then $\varphi \otimes \psi$ is a homomorphism of left S-modules. In particular, if R is commutative then $\varphi \otimes \psi$ is always an R-module homomorphism for the standard R-module structures.
(3) If $\lambda : M' \to M''$ and $\mu : N' \to N''$ are R-module homomorphisms then $(\lambda \otimes \mu) \circ (\varphi \otimes \psi) = (\lambda \circ \varphi) \otimes (\mu \circ \psi)$.

Proof: The map $(m, n) \mapsto \varphi(m) \otimes \psi(n)$ from $M \times N$ to $M' \otimes_R N'$ is clearly R-balanced, so (1) follows immediately from Theorem 10.

In (2) the definition of the (left) action of S on M together with the assumption that φ is an S-module homomorphism imply that on simple tensors

$$(\varphi \otimes \psi)(s(m \otimes n)) = (\varphi \otimes \psi)(sm \otimes n) = \varphi(sm) \otimes \psi(n) = s\varphi(m) \otimes \psi(n).$$

Since $\varphi \otimes \psi$ is additive, this extends to sums of simple tensors to show that $\varphi \otimes \psi$ is an S-module homomorphism. This gives (2).

The uniqueness condition in Theorem 10 implies (3), which completes the proof.

The next result shows that we may write $M \otimes N \otimes L$, or more generally, an n-fold tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$, unambiguously whenever it is defined.

Theorem 14. *(Associativity of the Tensor Product)* Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

of abelian groups such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. If M is an (S, R)-bimodule, then this is an isomorphism of S-modules.

Proof: Note first that the (R, T)-bimodule structure on N makes $M \otimes_R N$ into a right T-module and $N \otimes_T L$ into a left R-module, so both sides of the isomorphism are well defined. For each fixed $l \in L$, the mapping $(m, n) \mapsto m \otimes (n \otimes l)$ is R-balanced, so by Theorem 10 there is a homomorphism $M \otimes_R N \to M \otimes_R (N \otimes_T L)$ with $m \otimes n \mapsto m \otimes (n \otimes l)$. This shows that the map from $(M \otimes_R N) \times L$ to $M \otimes_R (N \otimes_T L)$ given by $(m \otimes n, l) \mapsto m \otimes (n \otimes l)$ is well defined. Since it is easily seen to be T-balanced, another application of Theorem 10 implies that it induces a homomorphism $(M \otimes_R N) \otimes_T L \to M \otimes_R (N \otimes_T L)$ such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. In a similar way we can construct a homomorphism in the opposite direction that is inverse to this one. This proves the group isomorphism.

Assume in addition M is an (S, R)-bimodule. Then for $s \in S$ and $t \in T$ we have

$$s\,((m \otimes n)t) = s(m \otimes nt) = sm \otimes nt = (sm \otimes n)t = (s(m \otimes n))\,t$$

so that $M \otimes_R N$ is an (S, T)-bimodule. Hence $(M \otimes_R N) \otimes_T L$ is a left S-module. Since $N \otimes_T L$ is a left R-module, also $M \otimes_R (N \otimes_T L)$ is a left S-module. The group isomorphism just established is easily seen to be a homomorphism of left S-modules by the same arguments used in previous proofs: it is additive and is S-linear on simple tensors since $s\,((m \otimes n) \otimes l) = s(m \otimes n) \otimes l = (sm \otimes n) \otimes l$ maps to the element $sm \otimes (n \otimes l) = s\,(m \otimes (n \otimes l))$. The proof is complete.

Corollary 15. Suppose R is commutative and M, N, and L are left R-modules. Then

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$$

as R-modules for the standard R-module structures on M, N and L.

There is a natural extension of the notion of a bilinear map:

Definition. Let R be a commutative ring with 1 and let $M_1, M_2, \ldots, M_n$ and L be R-modules with the standard R-module structures. A map $\varphi : M_1 \times \cdots \times M_n \to L$ is called *n-multilinear over R* (or simply *multilinear* if n and R are clear from the context) if it is an R-module homomorphism in each component when the other component entries are kept constant, i.e., for each i

$$\varphi(m_1, \ldots, m_{i-1}, rm_i + r'm_i', m_{i+1}, \ldots, m_n)$$
$$= r\varphi(m_1, \ldots, m_i, \ldots, m_n) + r'\varphi(m_1, \ldots, m_i', \ldots, m_n)$$

for all $m_i, m_i' \in M_i$ and $r, r' \in R$. When $n = 2$ (respectively, 3) one says φ is *bilinear* (respectively *trilinear*) rather than 2-multilinear (or 3-multilinear).

One may construct the n-fold tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ from first principles and prove its analogous universal property with respect to multilinear maps from $M_1 \times \cdots \times M_n$ to L. By the previous theorem and corollary, however, an n-fold tensor product may be obtained unambiguously by iterating the tensor product of pairs of modules since any bracketing of $M_1 \otimes \cdots \otimes M_n$ into tensor products of pairs gives an isomorphic R-module. The universal property of the tensor product of a pair of modules in Theorem 10 and Corollary 12 then implies that multilinear maps factor uniquely through the R-module $M_1 \otimes \cdots \otimes M_n$, i.e., this tensor product is the universal object with respect to multilinear functions:

Corollary 16. Let R be a commutative ring and let $M_1, \ldots, M_n, L$ be R-modules. Let $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ denote any bracketing of the tensor product of these modules and let

$$\iota : M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n$$

be the map defined by $\iota(m_1, \ldots, m_n) = m_1 \otimes \cdots \otimes m_n$. Then

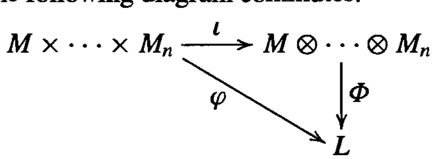
(1) for every R-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$ the map $\varphi = \Phi \circ \iota$ is n-multilinear from $M_1 \times \cdots \times M_n$ to L, and

(2) if $\varphi : M_1 \times \cdots \times M_n \to L$ is an n-multilinear map then there is a unique R-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$ such that $\varphi = \Phi \circ \iota$.

Hence there is a bijection

$$\left\{ \begin{array}{c} n\text{-multilinear maps} \\ \varphi : M_1 \times \cdots \times M_n \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R\text{-module homomorphisms} \\ \Phi : M_1 \otimes \cdots \otimes M_n \to L \end{array} \right\}$$

with respect to which the following diagram commutes:



We have already seen examples where $M_1 \otimes_R N$ is not contained in $M \otimes_R N$ even when M_1 is an R-submodule of M. The next result shows in particular that (an isomorphic copy of) $M_1 \otimes_R N$ *is* contained in $M \otimes_R N$ if M_1 is an R-module *direct summand* of M.

Theorem 17. *(Tensor Products of Direct Sums)* Let M, M' be right R-modules and let N, N' be left R-modules. Then there are unique group isomorphisms

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$
$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively. If M, M' are also (S, R)-bimodules, then these are isomorphisms of left S-modules. In particular, if R is commutative, these are isomorphisms of R-modules.

Proof: The map $(M \oplus M') \times N \to (M \otimes_R N) \oplus (M' \otimes_R N)$ defined by $((m, m'), n) \mapsto (m \otimes n, m' \otimes n)$ is well defined since m and m' in $M \oplus M'$ are uniquely defined in the direct sum. The map is clearly R-balanced, so induces a homomorphism f from $(M \oplus M') \otimes N$ to $(M \otimes_R N) \oplus (M' \otimes_R N)$ with

$$f((m, m') \otimes n) = (m \otimes n, m' \otimes n).$$

In the other direction, the R-balanced maps $M \times N \to (M \oplus M') \otimes_R N$ and $M' \times N \to (M \oplus M') \otimes_R N$ given by $(m, n) \mapsto (m, 0) \otimes n$ and $(m', n) \mapsto (0, m') \otimes n$, respectively, define homomorphisms from $M \otimes_R N$ and $M' \otimes_R N$ to $(M \oplus M') \otimes_R N$. These in turn give a homomorphism g from the direct sum $(M \otimes_R N) \oplus (M' \otimes_R N)$ to $(M \oplus M') \otimes_R N$ with

$$g((m \otimes n_1, m' \otimes n_2)) = (m, 0) \otimes n_1 + (0, m') \otimes n_2.$$

An easy check shows that f and g are inverse homomorphisms and are S-module isomorphisms when M and M' are (S, R)-bimodules. This completes the proof.

The previous theorem clearly extends by induction to any finite direct sum of R-modules. The corresponding result is also true for arbitrary direct sums. For example

$$M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} (M \otimes N_i),$$

where I is any index set (cf. the exercises). This result is referred to by saying that *tensor products commute with direct sums.*

Corollary 18. *(Extension of Scalars for Free Modules)* The module obtained from the free R-module $N \cong R^n$ by extension of scalars from R to S is the free S-module S^n, i.e.,

$$S \otimes_R R^n \cong S^n$$

as left S-modules.

Proof: This follows immediately from Theorem 17 and the isomorphism $S \otimes_R R \cong S$ proved in Example 7 previously.

Corollary 19. Let R be a commutative ring and let $M \cong R^s$ and $N \cong R^t$ be free R-modules with bases $m_1, \dots, m_s$ and $n_1, \dots, n_t$, respectively. Then $M \otimes_R N$ is a free R-module of rank st, with basis $m_i \otimes n_j$, $1 \le i \le s$ and $1 \le j \le t$, i.e.,

$$R^s \otimes_R R^t \cong R^{st}.$$

Remark: More generally, the tensor product of two free modules of arbitrary rank over a commutative ring is free (cf. the exercises).

Proof: This follows easily from Theorem 17 and the first example following Corollary 9.

Proposition 20. Suppose R is a commutative ring and M, N are left R-modules, considered with the standard R-module structures. Then there is a unique R-module isomorphism

$$M \otimes_R N \cong N \otimes_R M$$

mapping $m \otimes n$ to $n \otimes m$.

Proof: The map $M \times N \to N \otimes M$ defined by $(m, n) \mapsto n \otimes m$ is R-balanced. Hence it induces a unique homomorphism f from $M \otimes N$ to $N \otimes M$ with $f(m \otimes n) = n \otimes m$. Similarly, we have a unique homomorphism g from $N \otimes M$ to $M \otimes N$ with $g(n \otimes m) = m \otimes n$ giving the inverse of f, and both maps are easily seen to be R-module isomorphisms.

Remark: When $M = N$ it is not in general true that $a \otimes b = b \otimes a$ for $a, b \in M$. We shall study "symmetric tensors" in Section 11.6.

We end this section by showing that the tensor product of R-algebras is again an R-algebra.

Proposition 21. Let R be a commutative ring and let A and B be R-algebras. Then the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ is well defined and makes $A \otimes_R B$ into an R-algebra.

Proof: Note first that the definition of an R-algebra shows that

$$r(a \otimes b) = ra \otimes b = ar \otimes b = a \otimes rb = a \otimes br = (a \otimes b)r$$

for every $r \in R$, $a \in A$ and $b \in B$. To show that $A \otimes B$ is an R-algebra the main task is, as usual, showing that the specified multiplication is well defined. One way to proceed is to use two applications of Corollary 16, as follows. The map $\varphi : A \times B \times A \times B \to A \otimes B$ defined by $f(a, b, a', b') = aa' \otimes bb'$ is multilinear over R. For example,

$$\begin{aligned} f(a, r_1 b_1 + r_2 b_2, a', b') &= aa' \otimes (r_1 b_1 + r_2 b_2) b' \\ &= aa' \otimes r_1 b_1 b' + aa' \otimes r_2 b_2 b' \\ &= r_1 f(a, b_1, a', b') + r_2 f(a, b_2, a', b'). \end{aligned}$$

By Corollary 16, there is a corresponding R-module homomorphism Φ from $A \otimes B \otimes A \otimes B$ to $A \otimes B$ with $\Phi(a \otimes b \otimes a' \otimes b') = aa' \otimes bb'$. Viewing $A \otimes B \otimes A \otimes B$ as $(A \otimes B) \otimes (A \otimes B)$, we can apply Corollary 16 once more to obtain a well defined R-bilinear mapping φ' from $(A \otimes B) \times (A \otimes B)$ to $A \otimes B$ with $\varphi'(a \otimes b, a' \otimes b') = aa' \otimes bb'$. This shows that the multiplication is indeed well defined (and also that it satisfies the distributive laws). It is now a simple matter (left to the exercises) to check that with this multiplication $A \otimes B$ is an R-algebra.

Example

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is free of rank 4 as a module over $\mathbb{R}$ with basis given by $e_1 = 1 \otimes 1$, $e_2 = 1 \otimes i$, $e_3 = i \otimes 1$, and $e_4 = i \otimes i$ (by Corollary 19). By Proposition 21, this tensor product is also a (commutative) ring with $e_1 = 1$, and, for example,

$$e_4^2 = (i \otimes i)(i \otimes i) = i^2 \otimes i^2 = (-1) \otimes (-1) = (-1)(-1) \otimes 1 = 1.$$

Then $(e_4 - 1)(e_4 + 1) = 0$, so $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain.

The ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$-algebra and the left and right $\mathbb{R}$-actions are the same: $xr = rx$ for every $r \in \mathbb{R}$ and $x \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. The ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has a structure of a left $\mathbb{C}$-module because the first $\mathbb{C}$ is a $(\mathbb{C}, \mathbb{R})$-bimodule. It also has a right $\mathbb{C}$-module structure because the second $\mathbb{C}$ is an $(\mathbb{R}, \mathbb{C})$-bimodule. For example,

$$i \cdot e_1 = i \cdot (1 \otimes 1) = (i \cdot 1) \otimes 1 = i \otimes 1 = e_3$$

and

$$e_1 \cdot i = (1 \otimes 1) \cdot i = 1 \otimes (1 \cdot i) = 1 \otimes i = e_2.$$

This example also shows that even when the rings involved are commutative there may be natural left and right module structures (over some ring) that are not the same.

EXERCISES

Let R be a ring with 1.

1. Let $f : R \to S$ be a ring homomorphism from the ring R to the ring S with $f(1_R) = 1_S$. Verify the details that $sr = sf(r)$ defines a right R-action on S under which S is an (S, R)-bimodule.
2. Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.
3. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.
4. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]
5. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.
6. If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.
7. If R is any integral domain with quotient field Q and N is a left R-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and some $n \in N$.
8. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $U = R^\times$ be the set of nonzero elements in R and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements (u, n) with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n)$ if and only if $u'n = un'$ in N.

(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Section 4 of Chapter 15, cf. also Section 5 in Chapter 7.]

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R$, $b \in U$, $n \in N$, is an R-balanced map, so induces a homomorphism f from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u, n)}) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.

(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

(d) If A is an abelian group, show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ if and only if A is a torsion abelian group (i.e., every element of A has finite order).

9. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $Q \otimes_R N$ be the module obtained from N by extension of scalars from R to Q. Prove that the kernel of the R-module homomorphism $\iota : N \to Q \otimes_R N$ is the torsion submodule of N (cf. Exercise 8 in Section 1). [Use the previous exercise.]

10. Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \dots, e_n$.

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^{n} m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^{n} m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \dots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}$, $n = 1$, $M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

12. Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.

13. Prove that the usual dot product of vectors defined by letting $(a_1, \dots, a_n) \cdot (b_1, \dots, b_n)$ be $a_1b_1 + \cdots + a_nb_n$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.

14. Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-module. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I}(M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 20, Section 3. [Use the same argument as for the direct sum of two modules, taking care to note where the direct *sum* hypothesis is needed — cf. the next exercise.]

15. Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \dots$]

16. Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.

17. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R-module annihilated by both 2 and x.

(a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$\varphi(a_0 + a_1 x + \cdots + a_n x^n, b_0 + b_1 x + \cdots + b_m x^m) = \frac{a_0}{2} b_1 \bmod 2$$

is R-bilinear.

(b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2} q'(0)$ where q' denotes the usual polynomial derivative of q.

(c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

18. Suppose I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements (i.e., $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies that $m = 0$).

19. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$ as in Exercise 17. Show that the nonzero element $2 \otimes x - x \otimes 2$ in $I \otimes_R I$ is a torsion element. Show in fact that $2 \otimes x - x \otimes 2$ is annihilated by both 2 and x and that the submodule of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$ is isomorphic to R/I.

20. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.

21. Suppose R is commutative and let I and J be ideals of R.

(a) Show there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to the element ij.

(b) Give an example to show that the map in (a) need not be injective (cf. Exercise 17).

22. Suppose that M is a left and a right R-module such that $rm = mr$ for all $r \in R$ and $m \in M$. Show that the elements $r_1 r_2$ and $r_2 r_1$ act the same on M for every $r_1, r_2 \in R$. (This explains why the assumption that R is commutative in the definition of an R-algebra is a fairly natural one.)

23. Verify the details that the multiplication in Proposition 19 makes $A \otimes_R B$ into an R-algebra.

24. Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

25. Let R be a subring of the commutative ring S and let x be an indeterminate over S. Prove that $S[x]$ and $S \otimes_R R[x]$ are isomorphic as S-algebras.

26. Let S be a commutative ring containing R (with $1_S = 1_R$) and let $x_1, \ldots, x_n$ be independent indeterminates over the ring S. Show that for every ideal I in the polynomial ring $R[x_1, \ldots, x_n]$ that $S \otimes_R (R[x_1, \ldots, x_n]/I) \cong S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n]$ as S-algebras.

The next exercise shows the ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ introduced at the end of this section is isomorphic to $\mathbb{C} \times \mathbb{C}$. One may also prove this via Exercise 26 and Proposition 16 in Section 9.5, since $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$. The ring $\mathbb{C} \times \mathbb{C}$ is also discussed in Exercise 23 of Section 1.

27. (a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following Proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1 \epsilon_2 = 0$, $\epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called *orthogonal idempotents* in A). Deduce that A is isomorphic as a ring to the direct product of two principal ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \overline{z_2})$, where $\overline{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

10.5 EXACT SEQUENCES—PROJECTIVE, INJECTIVE, AND FLAT MODULES

One of the fundamental results for studying the structure of an algebraic object B (e.g., a group, a ring, or a module) is the First Isomorphism Theorem, which relates the subobjects of B (the normal subgroups, the ideals, or the submodules, respectively) with the possible homomorphic images of B. We have already seen many examples applying this theorem to understand the structure of B from an understanding of its "smaller" constituents—for example in analyzing the structure of the dihedral group D_8 by determining its center and the resulting quotient by the center.

In most of these examples we began *first* with a given B and then determined some of its basic properties by constructing a homomorphism φ (often given implicitly by the specification of $\ker\varphi \subseteq B$) and examining both $\ker\varphi$ and the resulting quotient $B/\ker\varphi$. We now consider in some greater detail the reverse situation, namely whether we may *first* specify the "smaller constituents." More precisely, we consider whether, given two modules A and C, there exists a module B containing (an isomorphic copy of) A such that the resulting quotient module B/A is isomorphic to C—in which case B is said to be an *extension of C by A*. It is then natural to ask how many such B exist for a given A and C, and the extent to which properties of B are determined by the corresponding properties of A and C. There are, of course, analogous problems in the contexts of groups and rings. This is the *extension problem* first discussed (for groups) in Section 3.4; in this section we shall be primarily concerned with left modules over a ring R, making note where necessary of the modifications required for some other structures, notably noncommutative groups. As in the previous section, throughout this section all rings contain a 1.

We first introduce a very convenient notation. To say that A is isomorphic to a submodule of B, is to say that there is an injective homomorphism $\psi : A \to B$ (so then $A \cong \psi(A) \subseteq B$). To say that C is isomorphic to the resulting quotient is to say that there is a surjective homomorphism $\varphi : B \to C$ with $\ker\varphi = \psi(A)$. In particular this gives us a pair of homomorphisms:

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} C$$

with image $\psi = \ker\varphi$. A pair of homomorphisms with this property is given a name:

Definition.

(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* (at Y) if $\text{image}\,\alpha = \ker\beta$.

(2) A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an *exact sequence* if it is exact at every X_n between a pair of homomorphisms.

With this terminology, the pair of homomorphisms $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ above is exact at B. We can also use this terminology to express the fact that for these maps ψ is injective and φ is surjective:

Proposition 22. Let A, B and C be R-modules over some ring R. Then

(1) The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.
(2) The sequence $B \xrightarrow{\varphi} C \to 0$ is exact (at C) if and only if φ is surjective.

Proof: The (uniquely defined) homomorphism $0 \to A$ has image 0 in A. This will be the kernel of ψ if and only if ψ is injective. Similarly, the kernel of the (uniquely defined) zero homomorphism $C \to 0$ is all of C, which is the image of φ if and only if φ is surjective.

Corollary 23. The sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is exact if and only if ψ is injective, φ is surjective, and image $\psi = \ker \varphi$, i.e., B is an extension of C by A.

Definition. The exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is called a *short exact sequence.*

In terms of this notation, the extension problem can be stated succinctly as follows: given modules A and C, determine all the short exact sequences

$$0 \longrightarrow A \xrightarrow{\ \psi\ } B \xrightarrow{\ \varphi\ } C \longrightarrow 0. \qquad (10.9)$$

We shall see below that the exact sequence notation is also extremely convenient for analyzing the extent to which properties of A and C determine the corresponding properties of B. If A, B and C are groups written multiplicatively, the sequence (9) will be written

$$1 \longrightarrow A \xrightarrow{\ \psi\ } B \xrightarrow{\ \varphi\ } C \longrightarrow 1 \qquad (10.9')$$

where 1 denotes the trivial group. Both Proposition 22 and Corollary 23 are valid with the obvious notational changes.

Note that any exact sequence can be written as a succession of short exact sequences since to say $X \xrightarrow{\ \alpha\ } Y \xrightarrow{\ \beta\ } Z$ is exact at Y is the same as saying that the sequence $0 \to \alpha(X) \to Y \to Y/\ker\beta \to 0$ is a short exact sequence.

Examples

(1) Given modules A and C we can always form their direct sum $B = A \oplus C$ and the sequence

$$0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$$

where $\iota(a) = (a, 0)$ and $\pi(a, c) = c$ is a short exact sequence. In particular, it follows that there always exists at least one extension of C by A.

(2) As a special case of the previous example, consider the two $\mathbb{Z}$-modules $A = \mathbb{Z}$ and $C = \mathbb{Z}/n\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\ \varphi\ } \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

giving one extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.

Another extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$ is given by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$$

where n denotes the map $x \mapsto nx$ given by multiplication by n, and π denotes the natural projection. Note that the modules in the middle of the previous two exact sequences are not isomorphic even though the respective "A" and "C" terms are isomorphic. Thus there are (at least) two "essentially different" or "inequivalent" ways of extending $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.

(3) If $\varphi : B \to C$ is any homomorphism we may form an exact sequence:

$$0 \longrightarrow \ker\varphi \xrightarrow{\iota} B \xrightarrow{\varphi} \text{image}\,\varphi \longrightarrow 0$$

where ι is the inclusion map. In particular, if φ is surjective, the sequence $\varphi : B \to C$ may be extended to a short exact sequence with $A = \ker\varphi$.

(4) One particularly important instance of the preceding example is when M is an R-module and S is a set of generators for M. Let $F(S)$ be the free R-module on S. Then

$$0 \longrightarrow K \xrightarrow{\iota} F(S) \xrightarrow{\varphi} M \longrightarrow 0$$

is the short exact sequence where φ is the unique R-module homomorphism which is the identity on S (cf. Theorem 6) and $K = \ker\varphi$.

More generally, when M is any group (possibly non-abelian) the above short exact sequence (with 1's at the ends, if M is written multiplicatively) describes a *presentation* of M, where K is the normal subgroup of $F(S)$ generated by the *relations* defining M (cf. Section 6.3).

(5) Two "inequivalent" extensions G of the Klein 4-group by the cyclic group Z_2 of order two are

$$1 \longrightarrow Z_2 \xrightarrow{\iota} D_8 \xrightarrow{\varphi} Z_2 \times Z_2 \longrightarrow 1, \text{ and}$$
$$1 \longrightarrow Z_2 \xrightarrow{\iota} Q_8 \xrightarrow{\varphi} Z_2 \times Z_2 \longrightarrow 1,$$

where in each case ι maps Z_2 injectively into the center of G (recall that both D_8 and Q_8 have centers of order two), and φ is the natural projection of G onto $G/Z(G)$.

Two other inequivalent extensions G of the Klein 4-group by Z_2 occur when G is either of the abelian groups $Z_2 \times Z_2 \times Z_2$ or $Z_2 \times Z_4$ for appropriate maps ι and φ.

Examples 2 and 5 above show that, for a fixed A and C, in general there may be several extensions of C by A. To distinguish different extensions we define the notion of a homomorphism (and isomorphism) between two exact sequences. Recall first that a diagram involving various homomorphisms is said to *commute* if any compositions of homomorphisms with the same starting and ending points are equal, i.e., the composite map defined by following a path of homomorphisms in the diagram depends only on the starting and ending points and not on the choice of the path taken.

Definition. Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules.

(1) A *homomorphism of short exact sequences* is a triple α, β, γ of module homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

The homomorphism is an *isomorphism of short exact sequences* if α, β, γ are all isomorphisms, in which case the extensions B and B' are said to be *isomorphic extensions*.

(2) The two exact sequences are called *equivalent* if $A = A'$, $C = C'$, and there is an isomorphism between them as in (1) that is the identity maps on A and C (i.e., α and γ are the identity). In this case the corresponding extensions B and B' are said to be *equivalent* extensions.

If B and B' are isomorphic extensions then in particular B and B' are isomorphic as R-modules, but more is true: there is an R-module isomorphism between B and B' that restricts to an isomorphism from A to A' and induces an isomorphism on the quotients C and C'. For a given A and C the condition that two extensions B and B' of C by A are equivalent is stronger still: there must exist an R-module isomorphism between B and B' that restricts to the *identity* map on A and induces the *identity* map on C. The notion of isomorphic extensions measures how many different extensions of C by A there are, allowing for C and A to be changed by an isomorphism. The notion of equivalent extensions measures how many different extensions of C by A there are when A and C are rigidly fixed.

Homomorphisms and isomorphisms between short exact sequences of multiplicative groups (9′) are defined similarly.

It is an easy exercise to see that the composition of homomorphisms of short exact sequences is also a homomorphism. Likewise, if the triple α, β, γ is an isomorphism (or equivalence) then $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ is an isomorphism (equivalence, respectively) in the reverse direction. It follows that "isomorphism" (or equivalence) is an equivalence relation on any set of short exact sequences.

Examples

(1) Let m and n be integers greater than 1. Assume n divides m and let $k = m/n$. Define a map from the exact sequence of $\mathbb{Z}$-modules in Example 2 of the preceding set of examples:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ n\ } & \mathbb{Z} & \xrightarrow{\ \pi\ } & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathbb{Z}/k\mathbb{Z} & \xrightarrow{\ \iota\ } & \mathbb{Z}/m\mathbb{Z} & \xrightarrow{\ \pi'\ } & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where α and β are the natural projections, γ is the identity map, ι maps $a \bmod k$ to $na \bmod m$, and π' is the natural projection of $\mathbb{Z}/m\mathbb{Z}$ onto its quotient $(\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z})$

(which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$). One easily checks that this is a homomorphism of short exact sequences.

(2) If again $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ is the short exact sequence of $\mathbb{Z}$-modules defined previously, map each module to itself by $x \mapsto -x$. This triple of homomorphisms gives an isomorphism of the exact sequence with itself. This isomorphism is not an equivalence of sequences since it is not the identity on the first $\mathbb{Z}$.

(3) The short exact sequences in Examples 1 and 2 following Corollary 23 are not isomorphic—the extension modules are not isomorphic $\mathbb{Z}$-modules (abelian groups). Likewise the two extensions, D_8 and Q_8, in Example 5 of the same set are not isomorphic (hence not equivalent), even though the two end terms "A" and "C" are the same for both sequences.

(4) Consider the maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\psi} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \beta & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\psi'} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\varphi'} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where ψ maps $\mathbb{Z}/2\mathbb{Z}$ injectively into the first component of the direct sum and φ projects the direct sum onto its second component. Also ψ' embeds $\mathbb{Z}/2\mathbb{Z}$ into the *second* component of the direct sum and φ' projects the direct sum onto its *first* component. If β maps the direct sum $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to itself by interchanging the two factors, then this diagram is seen to commute, hence giving an equivalence of the two exact sequences that is not the identity isomorphism.

(5) We exhibit two isomorphic but inequivalent $\mathbb{Z}$-module extensions. For $i = 1, 2$ define

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi_i} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $\psi : 1 \mapsto (2, 0)$ in both sequences, φ_1 is defined by $\varphi_1(a \bmod 4, b \bmod 2) = (a \bmod 2, b \bmod 2)$, and $\varphi_2(a \bmod 4, b \bmod 2) = (b \bmod 2, a \bmod 2)$. It is easy to see that the resulting two sequences are both short exact sequences.

An evident isomorphism between these two exact sequences is provided by the triple of maps id, id, γ, where $\gamma : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the map $\gamma((c, d)) = (d, c)$ that interchanges the two direct factors.

We now check that these two isomorphic sequences are *not equivalent*, as follows. Since $\varphi_1(0, 1) = (0, 1)$, any equivalence, id, β, id, from the first sequence to the second must map $(0, 1) \in \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to either $(1, 0)$ or $(3, 0)$ in $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, since these are the two possible elements mapping to $(0, 1)$ by φ_2. This is impossible, however, since the isomorphism β cannot send an element of order 2 to an element of order 4.

Put another way, equivalences involving the same extension module B are automorphisms of B that restrict to the identity on both $\psi(A)$ and $B/\psi(A)$. Any such automorphism of $B = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ must fix the coset $(0, 1) + \psi(A)$ since this is the unique nonidentity coset containing elements of order 2. Thus maps which send this coset to different elements in C give inequivalent extensions. In particular, there is yet a third inequivalent extension involving the same modules $A = \mathbb{Z}/2\mathbb{Z}$, $B = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $C = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, that maps the coset $(0, 1) + \psi(A)$ to the element $(1, 1) \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

By similar reasoning there are three inequivalent but isomorphic group extensions of $Z_2 \times Z_2$ by Z_2 with $B \cong D_8$ (cf. the exercises).

The homomorphisms α, β, γ in a homomorphism of short exact sequences are not independent. The next result gives some relations among these three homomorphisms.

Proposition 24. *(The Short Five Lemma)* Let α, β, γ be a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

(1) If α and γ are injective then so is β.
(2) If α and γ are surjective then so is β.
(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphic).

Remark: These results hold also for short exact sequences of (possibly non-abelian) groups (as the proof demonstrates).

Proof: We shall prove (1), leaving the proof of (2) as an exercise (and (3) follows immediately from (1) and (2)). Suppose then that α and γ are injective and suppose $b \in B$ with $\beta(b) = 0$. Let $\psi : A \to B$ and $\varphi : B \to C$ denote the homomorphisms in the first short exact sequence. Since $\beta(b) = 0$, it follows in particular that the image of $\beta(b)$ in the quotient C' is also 0. By the commutativity of the diagram this implies that $\gamma(\varphi(b)) = 0$, and since γ is assumed injective, we obtain $\varphi(b) = 0$, i.e., b is in the kernel of φ. By the exactness of the first sequence, this means that b is in the image of ψ, i.e., $b = \psi(a)$ for some $a \in A$. Then, again by the commutativity of the diagram, the image of $\alpha(a)$ in B' is the same as $\beta(\psi(a)) = \beta(b) = 0$. But α and the map from A' to B' are injective by assumption, and it follows that $a = 0$. Finally, $b = \psi(a) = \psi(0) = 0$ and we see that β is indeed injective.

We have already seen that there is always at least one extension of a module C by A, namely the direct sum $B = A \oplus C$. In this case the module B contains a submodule C' isomorphic to C (namely $C' = 0 \oplus C$) as well as the submodule A, and this submodule complement to A "splits" B into a direct sum. In the case of groups the existence of a subgroup complement C' to a normal subgroup in B implies that B is a semidirect product (cf. Section 5 in Chapter 5). The fact that B is a direct sum in the context of modules is a reflection of the fact that the underlying group structure in this case is *abelian*; for abelian groups semidirect products are direct products. In either case the corresponding short exact sequence is said to "split":

Definition.

(1) Let R be a ring and let $0 \to A \overset{\psi}{\to} B \overset{\varphi}{\to} C \to 0$ be a short exact sequence of R-modules. The sequence is said to be *split* if there is an R-module complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \oplus C$ (more precisely, $B = \psi(A) \oplus C'$ for some submodule C', and C' is mapped isomorphically onto C by φ: $\varphi(C') \cong C$).

(2) If $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ is a short exact sequence of groups, then the sequence is said to be *split* if there is a subgroup complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \rtimes C$ (more precisely, $B = \psi(A) \rtimes C'$ for some subgroup C', and C' is mapped isomorphically onto C by φ: $\varphi(C') \cong C$).

In either case the extension B is said to be a *split extension* of C by A.

The question of whether an extension splits is the question of the existence of a complement to $\psi(A)$ in B isomorphic (by φ) to C, so the notion of a split extension may equivalently be phrased in the language of homomorphisms:

Proposition 25. The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ of R-modules is split if and only if there is an R-module homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C. Similarly, the short exact sequence $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ of groups is split if and only if there is a group homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C.

Proof: This follows directly from the definitions: if μ is given define $C' = \mu(C) \subseteq B$ and if C' is given define $\mu = \varphi^{-1} : C \cong C' \subseteq B$.

Definition. With notation as in Proposition 25, any set map $\mu : C \to B$ such that $\varphi \circ \mu = \text{id}$ is called a *section* of φ. If μ is a *homomorphism* as in Proposition 25 then μ is called a *splitting homomorphism* for the sequence.

Note that a section of φ is nothing more than a choice of coset representatives in B for the quotient $B/\ker\varphi \cong C$. A section is a (splitting) homomorphism if this set of coset representatives forms a *submodule* (respectively, *subgroup*) in B, in which case this submodule (respectively, subgroup) gives a complement to $\psi(A)$ in B.

Examples

(1) The split short exact sequence $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$ has the evident splitting homomorphism $\mu(c) = (0, c)$.

(2) The extension $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} \to 0$, of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$ is split (with splitting homomorphism μ mapping $\mathbb{Z}/n\mathbb{Z}$ isomorphically onto the second factor of the direct sum). On the other hand, the exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ is not split since there is no nonzero homomorphism of $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Z}$.

(3) Neither D_8 nor Q_8 is a split extension of $Z_2 \times Z_2$ by Z_2 because in neither group is there a subgroup complement to the center (Section 2.5 gives the subgroup structures of these groups).

(4) The group D_8 *is* a split extension of Z_2 by Z_4, i.e., there is a split short exact sequence

$$1 \longrightarrow Z_4 \xrightarrow{\ \iota\ } D_8 \xrightarrow{\ \pi\ } Z_2 \longrightarrow 1,$$

namely,

$$1 \longrightarrow \langle r \rangle \xrightarrow{\ \iota\ } D_8 \xrightarrow{\ \pi\ } \langle \bar{s} \rangle \longrightarrow 1,$$

using our usual set of generators for D_8. Here ι is the inclusion map and $\pi : r^a s^b \mapsto \bar{s}^b$ is the projection onto the quotient $D_8/\langle r \rangle \cong Z_2$. The splitting homomorphism μ

maps $\langle \bar{s} \rangle$ isomorphically onto the complement $\langle s \rangle$ for $\langle r \rangle$ in D_8. Equivalently, D_8 is the semidirect product of the normal subgroup $\langle r \rangle$ (isomorphic to Z_4) with $\langle s \rangle$ (isomorphic to Z_2).

On the other hand, while Q_8 is also an extension of Z_2 by Z_4 (for example, $\langle i \rangle \cong Z_4$ has quotient isomorphic to Z_2), Q_8 is *not* a split extension of Z_2 by Z_4: no cyclic subgroup of Q_8 of order 4 has a complement in Q_8.

Section 5.5 contains many more examples of split extensions of groups.

Proposition 25 shows that an extension B of C by A is a split extension if and only if there is a splitting homomorphism μ of the projection map $\varphi : B \to C$ from B to the quotient C. The next proposition shows in particular that for modules this is equivalent to the existence of a splitting homomorphism for ψ at the other end of the sequence.

Proposition 26. Let $0 \to A \overset{\psi}{\to} B \overset{\varphi}{\to} C \to 0$ be a short exact sequence of modules (respectively, $1 \to A \overset{\psi}{\to} B \overset{\varphi}{\to} C \to 1$ a short exact sequence of groups). Then $B = \psi(A) \oplus C'$ for some submodule C' of B with $\varphi(C') \cong C$ (respectively, $B = \psi(A) \times C'$ for some subgroup C' of B with $\varphi(C') \cong C$) if and only if there is a homomorphism $\lambda : B \to A$ such that $\lambda \circ \psi$ is the identity map on A.

Proof: This is similar to the proof of Proposition 25. If λ is given, define $C' = \ker \lambda \subseteq B$ and if C' is given define $\lambda : B = \psi(A) \oplus C' \to A$ by $\lambda((\psi(a), c')) = a$. Note that in this case $C' = \ker \lambda$ is *normal* in B, so that C' is a *normal* complement to $\psi(A)$ in B, which in turn implies that B is the *direct sum* of $\psi(A)$ and C' (cf. Theorem 9 of Section 5.4).

Proposition 26 shows that for general group extensions, the existence of a splitting homomorphism λ on the *left* end of the sequence is stronger than the condition that the extension splits: in this case the extension group is a *direct* product, and not just a *semidirect* product. The fact that these two notions are equivalent in the context of modules is again a reflection of the abelian nature of the underlying groups, where semidirect products are always direct products.

Modules and $Hom_R(D, \_\_)$

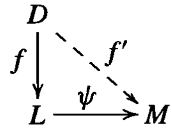
Let R be a ring with 1 and suppose the R-module M is an extension of N by L, with

$$0 \longrightarrow L \overset{\psi}{\longrightarrow} M \overset{\varphi}{\longrightarrow} N \longrightarrow 0$$

the corresponding short exact sequence of R-modules. It is natural to ask whether properties for L and N imply related properties for the extension M. The first situation we shall consider is whether an R-module homomorphism from some fixed R-module D to either L or N implies there is also an R-module homomorphism from D to M.

The question of obtaining a homomorphism from D to M given a homomorphism from D to L is easily disposed of: if $f \in \text{Hom}_R(D, L)$ is an R-module homomorphism from D to L then the composite $f' = \psi \circ f$ is an R-module homomorphism from D to

M. The relation between these maps can be indicated pictorially by the commutative diagram



Put another way, composition with ψ induces a map

$$\psi' : \mathrm{Hom}_R(D, L) \longrightarrow \mathrm{Hom}_R(D, M)$$
$$f \longmapsto f' = \psi \circ f.$$

Recall that, by Proposition 2, $\mathrm{Hom}_R(D, L)$ and $\mathrm{Hom}_R(D, M)$ are abelian groups.

Proposition 27. Let D, L and M be R-modules and let $\psi : L \to M$ be an R-module homomorphism. Then the map

$$\psi' : \mathrm{Hom}_R(D, L) \longrightarrow \mathrm{Hom}_R(D, M)$$
$$f \longmapsto f' = \psi \circ f$$

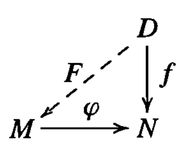
is a homomorphism of abelian groups. If ψ is injective, then ψ' is also injective, i.e.,

$$\text{if} \quad 0 \longrightarrow L \xrightarrow{\psi} M \quad \text{is exact,}$$

$$\text{then} \quad 0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \quad \text{is also exact.}$$

Proof: The fact that ψ' is a homomorphism is immediate. If ψ is injective, then distinct homomorphisms f and g from D into L give distinct homomorphisms $\psi \circ f$ and $\psi \circ g$ from D into M, which is to say that ψ' is also injective.

While obtaining homomorphisms into M from homomorphisms into the submodule L is straightforward, the situation for homomorphisms into the quotient N is much less evident. More precisely, given an R-module homomorphism $f : D \to N$ the question is whether there exists an R-module homomorphism $F : D \to M$ that *extends* or *lifts* f to M, i.e., that makes the following diagram commute:



As before, composition with the homomorphism φ induces a homomorphism of abelian groups

$$\varphi' : \mathrm{Hom}_R(D, M) \longrightarrow \mathrm{Hom}_R(D, N)$$
$$F \longmapsto F' = \varphi \circ F.$$

In terms of φ', the homomorphism f to N lifts to a homomorphism to M if and only if f is in the image of φ' (namely, f is the image of the lift F).

In general it may not be possible to lift a homomorphism f from D to N to a homomorphism from D to M. For example, consider the nonsplit exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ from the previous set of examples. Let $D = \mathbb{Z}/2\mathbb{Z}$ and let f be the identity map from D into N. Any homomorphism F of D into $M = \mathbb{Z}$ must map D to 0 (since $\mathbb{Z}$ has no elements of order 2), hence $\pi \circ F$ maps D to 0 in N, and in particular, $\pi \circ F \neq f$. Phrased in terms of the map φ', this shows that

$$\text{if} \quad M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

$$\text{then} \quad \operatorname{Hom}_R(D, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, N) \longrightarrow 0 \quad \text{is } \textit{not necessarily} \text{ exact.}$$

These results relating the homomorphisms into L and N to the homomorphisms into M can be neatly summarized as part of the following theorem.

Theorem 28. Let D, L, M, and N be R-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence

$$0 \to \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, N) \quad \text{is exact.} \qquad (10.10)$$

A homomorphism $f : D \to N$ lifts to a homomorphism $F : D \to M$ if and only if $f \in \operatorname{Hom}_R(D, N)$ is in the image of φ'. In general $\varphi' : \operatorname{Hom}_R(D, M) \to \operatorname{Hom}_R(D, N)$ need not be surjective; the map φ' is surjective if and only if every homomorphism from D to N lifts to a homomorphism from D to M, in which case the sequence (10) can be extended to a short exact sequence.

The sequence (10) is exact for *all* R-modules D if and only if the sequence

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \quad \text{is exact.}$$

Proof: The only item in the first statement that has not already been proved is the exactness of (10) at $\operatorname{Hom}_R(D, M)$, i.e., $\ker \varphi' = \operatorname{image} \psi'$. Suppose $F : D \to M$ is an element of $\operatorname{Hom}_R(D, M)$ lying in the kernel of φ', i.e., with $\varphi \circ F = 0$ as homomorphisms from D to N. If $d \in D$ is any element of D, this implies that $\varphi(F(d)) = 0$ and $F(d) \in \ker \varphi$. By the exactness of the sequence defining the extension M we have $\ker \varphi = \operatorname{image} \psi$, so there is some element $l \in L$ with $F(d) = \psi(l)$. Since ψ is injective, the element l is unique, so this gives a well defined map $F' : D \to L$ given by $F'(d) = l$. It is an easy check to verify that F' is a homomorphism, i.e., $F' \in \operatorname{Hom}_R(D, L)$. Since $\psi \circ F'(d) = \psi(l) = F(d)$, we have $F = \psi'(F')$ which shows that F is in the image of ψ', proving that $\ker \varphi' \subseteq \operatorname{image} \psi'$. Conversely, if F is in the image of ψ' then $F = \psi'(F')$ for some $F' \in \operatorname{Hom}_R(D, L)$ and so $\varphi(F(d)) = \varphi(\psi(F'(d)))$ for any $d \in D$. Since $\ker \varphi = \operatorname{image} \psi$ we have $\varphi \circ \psi = 0$, and it follows that $\varphi(F(d)) = 0$ for any $d \in D$, i.e., $\varphi'(F) = 0$. Hence F is in the kernel of φ', proving the reverse containment: $\operatorname{image} \psi' \subseteq \ker \varphi'$.

For the last statement in the theorem, note first that the surjectivity of φ was not required for the proof that (10) is exact, so the "if" portion of the statement has already been proved. For the converse, suppose that the sequence (10) is exact for all R-modules D. In general, $\mathrm{Hom}_R(R, X) \cong X$ for any left R-module X, the isomorphism being given by mapping a homomorphism to its value on the element $1 \in R$ (cf. Exercise 10(b)). Taking $D = R$ in (10), the exactness of the sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N$ follows easily.

By Theorem 28, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \longrightarrow 0 \qquad (10.11)$$

is in general *not* a short exact sequence since the homomorphism φ' need not be surjective. The question of whether this sequence is exact precisely measures the extent to which the homomorphisms from D into M are uniquely determined by pairs of homomorphisms from D into L and D into N. More precisely, this sequence is exact if and only if there is a bijection $F \leftrightarrow (g, f)$ between homomorphisms $F : D \to M$ and pairs of homomorphisms $g : D \to L$ and $f : D \to N$ given by $F|_{\psi(L)} = \psi'(g)$ and $f = \varphi'(F)$.

One situation in which the sequence (11) is exact occurs when the original sequence $0 \to L \to M \to N \to 0$ is a *split* exact sequence, i.e., when $M = L \oplus N$. In this case the sequence (11) is also a split exact sequence, as the first part of the following proposition shows.

Proposition 29. Let D, L and N be R-modules. Then
(1) $\mathrm{Hom}_R(D, L \oplus N) \cong \mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$, and
(2) $\mathrm{Hom}_R(L \oplus N, D) \cong \mathrm{Hom}_R(L, D) \oplus \mathrm{Hom}_R(N, D)$.

Proof: Let $\pi_1 : L \oplus N \to L$ be the natural projection from $L \oplus N$ to L and similarly let π_2 be the natural projection to N. If $f \in \mathrm{Hom}_R(D, L \oplus N)$ then the compositions $\pi_1 \circ f$ and $\pi_2 \circ f$ give elements in $\mathrm{Hom}_R(D, L)$ and $\mathrm{Hom}_R(D, N)$, respectively. This defines a map from $\mathrm{Hom}_R(D, L \oplus N)$ to $\mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$ which is easily seen to be a homomorphism. Conversely, given $f_1 \in \mathrm{Hom}_R(D, L)$ and $f_2 \in \mathrm{Hom}_R(D, N)$, define the map $f \in \mathrm{Hom}_R(D, L \oplus N)$ by $f(d) = (f_1(d), f_2(d))$. This defines a map from $\mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$ to $\mathrm{Hom}_R(D, L \oplus N)$ that is easily checked to be a homomorphism inverse to the map above, proving the isomorphism in (1). The proof of (2) is similar and is left as an exercise.

The results in Proposition 29 extend immediately by induction to any finite direct sum of R-modules. These results are referred to by saying that Hom *commutes with finite direct sums in either variable* (compare to Theorem 17 for a corresponding result for tensor products). For infinite direct sums the situation is more complicated. Part (1) remains true if $L \oplus N$ is replaced by an arbitrary direct sum and the direct sum on the right hand side is replaced by a direct product (Exercise 13 shows that the direct product is necessary). Part (2) remains true if the direct sums on both sides are replaced by direct products.

This proposition shows that if the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a split short exact sequence of R-modules, then

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \longrightarrow 0$$

is also a split short exact sequence of abelian groups for every R-module D. Exercise 14 shows that a converse holds: if $0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \to 0$ is exact for every R-module D then $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a split short exact sequence (which then implies that if the original Hom sequence is exact for every D, then in fact it is split exact for every D).

Proposition 29 identifies a situation in which the sequence (11) is exact in terms of the modules L, M, and N. The next result adopts a slightly different perspective, characterizing instead the modules D having the property that the sequence (10) in Theorem 28 can *always* be extended to a short exact sequence:

Proposition 30. Let P be an R-module. Then the following are equivalent:

(1) For any R-modules L, M, and N, if

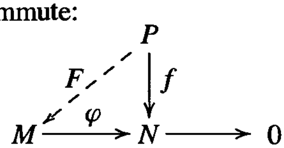
$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \xrightarrow{\psi'} \mathrm{Hom}_R(P, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

is also a short exact sequence.

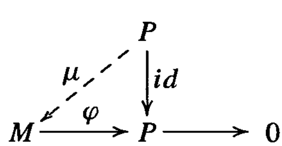
(2) For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M, i.e., given $f \in \mathrm{Hom}_R(P, N)$ there is a lift $F \in \mathrm{Hom}_R(P, M)$ making the following diagram commute:



(3) If P is a quotient of the R-module M then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits.

(4) P is a direct summand of a free R-module.

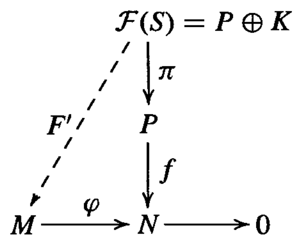
Proof: The equivalence of (1) and (2) is a restatement of a result in Theorem 28. Suppose now that (2) is satisfied, and let $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} P \to 0$ be exact. By (2), the identity map from P to P lifts to a homomorphism μ making the following diagram commute:



Then $\varphi \circ \mu = 1$, so μ is a splitting homomorphism for the sequence, which proves (3).

Every module P is the quotient of a free module (for example, the free module on the set of elements in P), so there is always an exact sequence $0 \to \ker\varphi \to \mathcal{F} \xrightarrow{\varphi} P \to 0$ where $\mathcal{F}$ is a free R-module (cf. Example 4 following Corollary 23). If (3) is satisfied, then this sequence splits, so $\mathcal{F}$ is isomorphic to the direct sum of $\ker\varphi$ and P, which proves (4).

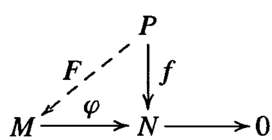
Finally, to prove (4) implies (2), suppose that P is a direct summand of a free R-module on some set S, say $\mathcal{F}(S) = P \oplus K$, and that we are given a homomorphism f from P to N as in (2). Let π denote the natural projection from $\mathcal{F}(S)$ to P, so that $f \circ \pi$ is a homomorphism from $\mathcal{F}(S)$ to N. For any $s \in S$ define $n_s = f \circ \pi(s) \in N$ and let $m_s \in M$ be any element of M with $\varphi(m_s) = n_s$ (which exists because φ is surjective). By the universal property for free modules (Theorem 6 of Section 3), there is a unique R-module homomorphism F' from $\mathcal{F}(S)$ to M with $F'(s) = m_s$. The diagram is the following:



By definition of the homomorphism F' we have $\varphi \circ F'(s) = \varphi(m_s) = n_s = f \circ \pi(s)$, from which it follows that $\varphi \circ F' = f \circ \pi$ on $\mathcal{F}(S)$, i.e., the diagram above is commutative. Now define a map $F : P \to M$ by $F(d) = F'((d, 0))$. Since F is the composite of the injection $P \to \mathcal{F}(S)$ with the homomorphism F', it follows that F is an R-module homomorphism. Then

$$\varphi \circ F(d) = \varphi \circ F'((d, 0)) = f \circ \pi((d, 0)) = f(d)$$

i.e., $\varphi \circ F = f$, so the diagram



commutes, which proves that (4) implies (2) and completes the proof.

Definition. An R-module P is called *projective* if it satisfies any of the equivalent conditions of Proposition 30.

The third statement in Proposition 30 can be rephrased as saying that any module M that projects onto P has (an isomorphic copy of) P as a direct summand, which explains the terminology.

The following result is immediate from Proposition 30 (and its proof):

Corollary 31. Free modules are projective. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module. Every module is a quotient of a projective module.

If D is fixed, then given any R-module X we have an associated abelian group $\text{Hom}_R(D, X)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha' : \text{Hom}_R(D, X) \to \text{Hom}_R(D, Y)$, defined by $\alpha'(f) = \alpha \circ f$. Put another way, the map $\text{Hom}_R(D, \_\_)$ is a *covariant functor* from the category of R-modules to the category of abelian groups (cf. Appendix II). Theorem 28 shows that applying this functor to the terms in the exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

produces an exact sequence

$$0 \to \text{Hom}_R(D, L) \xrightarrow{\psi'} \text{Hom}_R(D, M) \xrightarrow{\varphi'} \text{Hom}_R(D, N).$$

This is referred to by saying that $\text{Hom}_R(D, \_\_)$ is a *left exact* functor. By Proposition 30, the functor $\text{Hom}_R(D, \_\_)$ is *exact*, i.e., always takes short exact sequences to short exact sequences, if and only if D is projective. We summarize this as

Corollary 32. If D is an R-module, then the functor $\text{Hom}_R(D, \_\_)$ from the category of R-modules to the category of abelian groups is left exact. It is exact if and only if D is a projective R-module.

Note that if $\text{Hom}_R(D, \_\_)$ takes short exact sequences to short exact sequences, then it takes exact sequences of any length to exact sequences since any exact sequence can be broken up into a succession of short exact sequences.

As we have seen, the functor $\text{Hom}_R(D, \_\_)$ is in general not exact on the right. Measuring the extent to which functors such as $\text{Hom}_R(D, \_\_)$ fail to be exact leads to the notions of "homological algebra," considered in Chapter 17.

Examples

(1) We shall see in Section 11.1 that if $R = F$ is a field then every F-module is projective (although we only prove this for finitely generated modules).

(2) By Corollary 31, $\mathbb{Z}$ is a projective $\mathbb{Z}$-module. This can be seen directly as follows: suppose f is a map from $\mathbb{Z}$ to N and $M \xrightarrow{\varphi} N \to 0$ is exact. The homomorphism f is uniquely determined by the value $n = f(1)$. Then f can be lifted to a homomorphism $F : \mathbb{Z} \to M$ by first defining $F(1) = m$, where m is any element in M mapped to n by φ, and then extending F to all of $\mathbb{Z}$ by additivity.

By the first statement in Proposition 30, since $\mathbb{Z}$ is projective, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}$-modules, then

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, L) \xrightarrow{\psi'} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \xrightarrow{\varphi'} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \longrightarrow 0$$

is also an exact sequence. This can also be seen directly using the isomorphism $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \cong M$ of abelian groups, which shows that the two exact sequences above are essentially the same.

(3) Free $\mathbb{Z}$-modules have no nonzero elements of finite order so no nonzero finite abelian group can be isomorphic to a submodule of a free module. By Corollary 31 it follows that no nonzero finite abelian group is a projective $\mathbb{Z}$-module.

(4) As a particular case of the preceding example, we see that for $n \geq 2$ the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective. By Theorem 28 it must be possible to find a short exact sequence which after applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_\_)$ is no longer exact on the right. One such sequence is the exact sequence of Example 2 following Corollary 23:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

for $n \geq 2$. Note first that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ since there are no nonzero $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}$. It is also easy to see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, as follows. Every homomorphism f is uniquely determined by $f(1) = a \in \mathbb{Z}/n\mathbb{Z}$, and given any $a \in \mathbb{Z}/n\mathbb{Z}$ there is a unique homomorphism f_a with $f_a(1) = a$; the map $f_a \mapsto a$ is easily checked to be an isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ to $\mathbb{Z}/n\mathbb{Z}$.

Applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_\_)$ to the short exact sequence above thus gives the sequence

$$0 \longrightarrow 0 \xrightarrow{n'} 0 \xrightarrow{\pi'} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which is not exact at its only nonzero term.

(5) Since $\mathbb{Q}/\mathbb{Z}$ is a torsion $\mathbb{Z}$-module it is not a submodule of a free $\mathbb{Z}$-module, hence is not projective. Note also that the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ does not split since $\mathbb{Q}$ contains no submodule isomorphic to $\mathbb{Q}/\mathbb{Z}$.

(6) The $\mathbb{Z}$-module $\mathbb{Q}$ is not projective (cf. the exercises).

(7) We shall see in Chapter 12 that a finitely generated $\mathbb{Z}$-module is projective if and only if it is free.

(8) Let R be the commutative ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ under componentwise addition and multiplication. If P_1 and P_2 are the principal ideals generated by $(1, 0)$ and $(0, 1)$ respectively then $R = P_1 \oplus P_2$, hence both P_1 and P_2 are projective R-modules by Proposition 30. Neither P_1 nor P_2 is free, since any free module has order a multiple of four.

(9) The direct sum of two projective modules is again projective (cf. Exercise 3).

(10) We shall see in Part VI that if F is any field and $n \in \mathbb{Z}^+$ then the ring $R = M_n(F)$ of all $n \times n$ matrices with entries from F has the property that every R-module is projective. We shall also see that if G is a finite group of order n and $n \neq 0$ in the field F then the group ring FG also has the property that every module is projective.

Injective Modules and $\mathrm{Hom}_R(\_\_, D)$

If $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ is a short exact sequence of R-modules then, instead of considering maps *from* an R-module D into L or N and the extent to which these determine maps from D into M, we can consider the "dual" question of maps from L or N *to* D. In this case, it is easy to dispose of the situation of a map from N to D: an R-module map from N to D immediately gives a map from M to D simply by composing with φ. It is easy to check that this defines an injective homomorphism of abelian groups

$$\varphi' : \mathrm{Hom}_R(N, D) \longrightarrow \mathrm{Hom}_R(M, D)$$
$$f \longmapsto f' = f \circ \varphi,$$

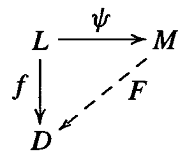
or, put another way,

$$\text{if} \quad M \xrightarrow{\varphi} N \to 0 \quad \text{is exact,}$$

$$\text{then} \quad 0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \quad \text{is exact.}$$

(Note that the associated maps on the homomorphism groups are in the reverse direction from the original maps.)

On the other hand, given an R-module homomorphism f from L to D it may not be possible to extend f to a map F from M to D, i.e., given f it may not be possible to find a map F making the following diagram commute:



For example, consider the exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ of $\mathbb{Z}$-modules, where ψ is multiplication by 2 and φ is the natural projection. Take $D = \mathbb{Z}/2\mathbb{Z}$ and let $f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be reduction modulo 2 on the first $\mathbb{Z}$ in the sequence. There is only one nonzero homomorphism F from the second $\mathbb{Z}$ in the sequence to $\mathbb{Z}/2\mathbb{Z}$ (namely, reduction modulo 2), but this F does not lift the map f since $F \circ \psi(\mathbb{Z}) = F(2\mathbb{Z}) = 0$, so $F \circ \psi \neq f$.

Composition with ψ induces an abelian group homomorphism ψ' from$\mathrm{Hom}_R(M, D)$ to $\mathrm{Hom}_R(L, D)$, and in terms of the map ψ', the homomorphism $f \in \mathrm{Hom}_R(L, D)$ can be lifted to a homomorphism from M to D if and only if f is in the image of ψ'. The example above shows that

$$\text{if} \quad 0 \longrightarrow L \xrightarrow{\psi} M \quad \text{is exact,}$$

$$\text{then} \quad \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \longrightarrow 0 \quad \text{is } \textit{not necessarily} \text{ exact.}$$

We can summarize these results in the following dual version of Theorem 28:

Theorem 33. Let D, L, M, and N be R-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence

$$0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \quad \text{is exact.} \tag{10.12}$$

A homomorphism $f : L \to D$ lifts to a homomorphism $F : M \to D$ if and only if $f \in \mathrm{Hom}_R(L, D)$ is in the image of ψ'. In general $\psi' : \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$ need not be surjective; the map ψ' is surjective if and only if every homomorphism from L to D lifts to a homomorphism from M to D, in which case the sequence (12) can be extended to a short exact sequence.

The sequence (12) is exact for *all* R-modules D if and only if the sequence

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \quad \text{is exact.}$$

Proof: The only item remaining to be proved in the first statement is the exactness of (12) at $\text{Hom}_R(M, D)$. The proof of this statement is very similar to the proof of the corresponding result in Theorem 28 and is left as an exercise. Note also that the injectivity of ψ is not required, which proves the "if" portion of the final statement of the theorem.

Suppose now that the sequence (12) is exact for all R-modules D. We first show that $\varphi : M \to N$ is a surjection. Take $D = N/\varphi(M)$. If $\pi_1 : N \to N/\varphi(M)$ is the natural projection homomorphism, then $\pi_1 \circ \varphi(M) = 0$ by definition of π_1. Since $\pi_1 \circ \varphi = \varphi'(\pi_1)$, this means that the element $\pi_1 \in \text{Hom}_R(N, N/\varphi(M))$ is mapped to 0 by φ'. Since φ' is assumed to be injective for all modules D, this means π_1 is the zero map, i.e., $N = \varphi(M)$ and so φ is a surjection. We next show that $\varphi \circ \psi = 0$, which will imply that image $\psi \subseteq \ker \varphi$. For this we take $D = N$ and observe that the identity map id_N on N is contained in $\text{Hom}_R(N, N)$, hence $\varphi'(id_N) \in \text{Hom}_R(M, N)$. Then the exactness of (12) for $D = N$ implies that $\varphi'(id_N) \in \ker \psi'$, so $\psi'(\varphi'(id_N)) = 0$. Then $id_N \circ \psi \circ \varphi = 0$, i.e., $\psi \circ \varphi = 0$, as claimed. Finally, we show that $\ker \varphi \subseteq$ image ψ. Let $D = M/\psi(L)$ and let $\pi_2 : M \to M/\psi(L)$ be the natural projection. Then $\psi'(\pi_2) = 0$ since $\pi_2(\psi(L)) = 0$ by definition of π_2. The exactness of (12) for this D then implies that π_2 is in the image of φ', say $\pi_2 = \varphi'(f)$ for some homomorphism $f \in \text{Hom}_R(N, M/\psi(L))$, i.e., $\pi_2 = f \circ \varphi$. If $m \in \ker \varphi$ then $\pi_2(m) = f(\varphi(m)) = 0$, which means that $m \in \psi(L)$ since π_2 is just the projection from M into the quotient $M/\psi(L)$. Hence $\ker \varphi \subseteq$ image ψ, completing the proof.

By Theorem 33, the sequence

$$0 \longrightarrow \text{Hom}_R(N, D) \xrightarrow{\varphi'} \text{Hom}_R(M, D) \xrightarrow{\psi'} \text{Hom}_R(L, D) \longrightarrow 0$$

is in general *not* a short exact sequence since ψ' need not be surjective, and the question of whether this sequence is exact precisely measures the extent to which homomorphisms from M to D are uniquely determined by pairs of homomorphisms from L and N to D.

The second statement in Proposition 29 shows that this sequence is exact when the original exact sequence $0 \to L \to M \to N \to 0$ is a *split* exact sequence. In fact in this case the sequence $0 \to \text{Hom}_R(N, D) \xrightarrow{\varphi'} \text{Hom}_R(M, D) \xrightarrow{\psi'} \text{Hom}_R(L, D) \to 0$ is also a split exact sequence of abelian groups for every R-module D. Exercise 14 shows that a converse holds: if $0 \to \text{Hom}_R(N, D) \xrightarrow{\varphi'} \text{Hom}_R(M, D) \xrightarrow{\psi'} \text{Hom}_R(L, D) \to 0$ is exact for every R-module D then $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a split short exact sequence (which then implies that if the Hom sequence is exact for every D, then in fact it is split exact for every D).

There is also a dual version of the first three parts of Proposition 30, which describes the R-modules D having the property that the sequence (12) in Theorem 33 can *always* be extended to a short exact sequence:

Proposition 34. Let Q be an R-module. Then the following are equivalent:

(1) For any R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_R(N, Q) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, Q) \xrightarrow{\psi'} \mathrm{Hom}_R(L, Q) \longrightarrow 0$$

is also a short exact sequence.

(2) For any R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is exact, then every R-module homomorphism from L into Q lifts to an R-module homomorphism of M into Q, i.e., given $f \in \mathrm{Hom}_R(L, Q)$ there is a lift $F \in \mathrm{Hom}_R(M, Q)$ making the following diagram commute:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\psi} & M \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle F} & \\ & & Q & & \end{array}$$

(3) If Q is a submodule of the R-module M then Q is a direct summand of M, i.e., every short exact sequence $0 \to Q \to M \to N \to 0$ splits.

Proof: The equivalence of (1) and (2) is part of Theorem 33. Suppose now that (2) is satisfied and let $0 \to Q \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ be exact. Taking $L = Q$ and f the identity map from Q to itself, it follows by (2) that there is a homomorphism $F : M \to Q$ with $F \circ \psi = 1$, so F is a splitting homomorphism for the sequence, which proves (3). The proof that (3) implies (2) is outlined in the exercises.

Definition. An R-module Q is called *injective* if it satisfies any of the equivalent conditions of Proposition 34.

The third statement in Proposition 34 can be rephrased as saying that any module M into which Q injects has (an isomorphic copy of) Q as a direct summand, which explains the terminology.

If D is fixed, then given any R-module X we have an associated abelian group $\mathrm{Hom}_R(X, D)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha' : \mathrm{Hom}_R(Y, D) \to \mathrm{Hom}_R(X, D)$, defined by $\alpha'(f) = f \circ \alpha$, that "reverses" the direction of the arrow. Put another way, the map $\mathrm{Hom}_R(D, \_\_)$ is a *contravariant functor* from the category of R-modules to the category of abelian groups (cf. Appendix II). Theorem 33 shows that applying this functor to the terms in the exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

produces an exact sequence

$$0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D).$$

This is referred to by saying that $\mathrm{Hom}_R(\_\_, D)$ is a *left exact* (contravariant) functor. Note that the functor $\mathrm{Hom}_R(\_\_, D)$ and the functor $\mathrm{Hom}_R(D, \_\_)$ considered earlier are both left exact; the former reverses the directions of the maps in the original short exact sequence, the latter maintains the directions of the maps.

By Proposition 34, the functor $\mathrm{Hom}_R(\_\_, D)$ is *exact*, i.e., always takes short exact sequences to short exact sequences (and hence exact sequences of any length to exact sequences), if and only if D is injective. We summarize this in the following proposition, which is dual to the covariant result of Corollary 32.

Corollary 35. If D is an R-module, then the functor $\mathrm{Hom}_R(\_\_, D)$ from the category of R-modules to the category of abelian groups is left exact. It is exact if and only if D is an injective R-module.

We have seen that an R-module is projective if and only if it is a direct summand of a free R-module. Providing such a simple characterization of injective R-modules is not so easy. The next result gives a criterion for Q to be an injective R-module (a result due to Baer, who introduced the notion of injective modules around 1940), and using it we can give a characterization of injective modules when $R = \mathbb{Z}$ (or, more generally, when R is a P.I.D.). Recall that a $\mathbb{Z}$-module A (i.e., an abelian group, written additively) is said to be *divisible* if $A = nA$ for all nonzero integers n. For example, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible (cf. Exercises 18 and 19 in Section 2.4 and Exercise 15 in Section 3.1).

Proposition 36. Let Q be an R-module.

(1) *(Baer's Criterion)* The module Q is injective if and only if for every left ideal I of R any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.

(2) If R is a P.I.D. then Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$. In particular, a $\mathbb{Z}$-module is injective if and only if it is divisible. When R is a P.I.D., quotient modules of injective R-modules are again injective.

Proof: If Q is injective and $g : I \to Q$ is an R-module homomorphism from the nonzero ideal I of R into Q, then g can be extended to an R-module homomorphism from R into Q by Proposition 34(2) applied to the exact sequence $0 \to I \to R$, which proves the "only if" portion of (1). Suppose conversely that every homomorphism $g : I \to Q$ can be lifted to a homomorphism $G : R \to Q$. To show that Q is injective we must show that if $0 \to L \to M$ is exact and $f : L \to Q$ is an R-module homomorphism then there is a lift $F : M \to Q$ extending f. If $\mathcal{S}$ is the collection (f', L') of lifts $f' : L' \to Q$ of f to a submodule L' of M containing L, then the ordering $(f', L') \leq (f'', L'')$ if $L' \subseteq L''$ and $f'' = f'$ on L' partially orders $\mathcal{S}$. Since $\mathcal{S} \neq \emptyset$, by Zorn's Lemma there is a maximal element (F, M') in $\mathcal{S}$. The map $F : M' \to Q$ is a lift of f and it suffices to show that $M' = M$. Suppose that there is some element $m \in M$ not contained in M' and let $I = \{r \in R \mid rm \in M'\}$. It is easy to check that I is a left ideal in R, and the map $g : I \to Q$ defined by $g(x) = F(xm)$ is an R-module homomorphism from I to Q. By hypothesis, there is a lift $G : R \to Q$ of g. Consider the submodule $M' + Rm$ of M, and define the map $F' : M' + Rm \to Q$ by $F'(m' + rm) = F(m') + G(r)$. If $m_1 + r_1 m = m_2 + r_2 m$ then $(r_1 - r_2)m = m_2 - m_1$

shows that $r_1 - r_2 \in I$, so that

$$G(r_1 - r_2) = g(r_1 - r_2) = F((r_1 - r_2)m) = F(m_2 - m_1),$$

and so $F(m_1) + G(r_1) = F(m_2) + G(r_2)$. Hence F' is well defined and it is then immediate that F' is an R-module homomorphism extending f to $M' + Rm$. This contradicts the maximality of M', so that $M' = M$, which completes the proof of (1).

To prove (2), suppose R is a P.I.D. Any nonzero ideal I of R is of the form $I = (r)$ for some nonzero element r of R. An R-module homomorphism $f : I \to Q$ is completely determined by the image $f(r) = q$ in Q. This homomorphism can be extended to a homomorphism $F : R \to Q$ if and only if there is an element q' in Q with $F(1) = q'$ satisfying $q = f(r) = F(r) = rq'$. It follows that Baer's criterion for Q is satisfied if and only if $rQ = Q$, which proves the first two statements in (2). The final statement follows since a quotient of a module Q with $rQ = Q$ for all $r \neq 0$ in R has the same property.

Examples

(1) Since $\mathbb{Z}$ is not divisible, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. This also follows from the fact that the exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ corresponding to multiplication by 2 does not split.

(2) The rational numbers $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

(3) The quotient $\mathbb{Q}/\mathbb{Z}$ of the injective $\mathbb{Z}$-module $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

(4) It is immediate that a direct sum of divisible $\mathbb{Z}$-modules is again divisible, hence a direct sum of injective $\mathbb{Z}$-modules is again injective. For example, $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. (See also Exercise 4).

(5) We shall see in Chapter 12 that no nonzero finitely generated $\mathbb{Z}$-module is injective.

(6) Suppose that the ring R is an integral domain. An R-module A is said to be a *divisible* R-module if $rA = A$ for every nonzero $r \in R$. The proof of Proposition 36 shows that in this case an injective R-module is divisible.

(7) We shall see in Section 11.1 that if $R = F$ is a field then every F-module is injective.

(8) We shall see in Part VI that if F is any field and $n \in \mathbb{Z}^+$ then the ring $R = M_n(F)$ of all $n \times n$ matrices with entries from F has the property that every R-module is injective (and also projective). We shall also see that if G is a finite group of order n and $n \neq 0$ in the field F then the group ring FG also has the property that every module is injective (and also projective).

Corollary 37. Every $\mathbb{Z}$-module is a submodule of an injective $\mathbb{Z}$-module.

Proof: Let M be a $\mathbb{Z}$-module and let A be any set of $\mathbb{Z}$-module generators of M. Let $\mathcal{F} = F(A)$ be the free $\mathbb{Z}$-module on the set A. Then by Theorem 6 there is a surjective $\mathbb{Z}$-module homomorphism from $\mathcal{F}$ to M and if $\mathcal{K}$ denotes the kernel of this homomorphism then $\mathcal{K}$ is a $\mathbb{Z}$-submodule of $\mathcal{F}$ and we can identify $M = \mathcal{F}/\mathcal{K}$. Let $\mathcal{Q}$ be the free $\mathbb{Q}$-module on the set A. Then $\mathcal{Q}$ is a direct sum of a number of copies of $\mathbb{Q}$, so is a divisible, hence (by Proposition 36) injective, $\mathbb{Z}$-module containing $\mathcal{F}$. Then $\mathcal{K}$ is also a $\mathbb{Z}$-submodule of $\mathcal{Q}$, so the quotient $\mathcal{Q}/\mathcal{K}$ is injective, again by Proposition 36. Since $M = \mathcal{F}/\mathcal{K} \subseteq \mathcal{Q}/\mathcal{K}$, it follows that M is contained in an injective $\mathbb{Z}$-module.

Corollary 37 can be used to prove the following more general version valid for arbitrary R-modules. This theorem is the injective analogue of the results in Theorem 6 and Corollary 31 showing that every R-module is a quotient of a projective R-module.

Theorem 38. Let R be a ring with 1 and let M be an R-module. Then M is contained in an injective R-module.

Proof: A proof is outlined in Exercises 15 to 17.

It is possible to prove a sharper result than Theorem 38, namely that there is a *minimal* injective R-module H containing M in the sense that any injective map of M into an injective R-module Q factors through H. More precisely, if $M \subseteq Q$ for an injective R-module Q then there is an injection $\iota : H \hookrightarrow Q$ that restricts to the identity map on M; using ι to identify H as a subset of Q we have $M \subseteq H \subseteq Q$. (cf. Theorem 57.13 in *Representation Theory of Finite Groups and Associative Algebras* by C. Curtis and I. Reiner, John Wiley & Sons, 1966). This module H is called the *injective hull* or *injective envelope* of M. The universal property of the injective hull of M with respect to inclusions of M into injective R-modules should be compared to the universal property with respect to homomorphisms of M of the free module $F(A)$ on a set of generators A for M in Theorem 6. For example, the injective hull of $\mathbb{Z}$ is $\mathbb{Q}$, and the injective hull of any field is itself (cf. the exercises).

Flat Modules and $D \otimes_R$ __

We now consider the behavior of extensions $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ of R-modules with respect to tensor products.

Suppose that D is a *right* R-module. For any homomorphism $f : X \to Y$ of left R-modules we obtain a homomorphism $1 \otimes f : D \otimes_R X \to D \otimes_R Y$ of abelian groups (Theorem 13). If in addition D is an (S, R)-bimodule (for example, when $S = R$ is commutative and D is given the standard (R, R)-bimodule structure as in Section 4), then $1 \otimes f$ is a homomorphism of left S-modules. Put another way,

$$D \otimes_R \_\_ : X \longrightarrow D \otimes_R X$$

is a *covariant functor* from the category of left R-modules to the category of abelian groups (respectively, to the category of left S-modules when D is an (S, R)-bimodule), cf. Appendix II. In a similar way, if D is a left R-module then $\_\_ \otimes_R D$ is a covariant functor from the category of right R-modules to the category of abelian groups (respectively, to the category of right S-modules when D is an (R, S)-bimodule). Note that, unlike Hom, the tensor product is covariant in both variables, and we shall therefore concentrate on $D \otimes_R \_\_$, leaving as an exercise the minor alterations necessary for $\_\_ \otimes_R D$.

We have already seen examples where the map $1 \otimes \psi : D \otimes_R L \to D \otimes_R M$ induced by an injective map $\psi : L \hookrightarrow M$ is no longer injective (for example the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ of $\mathbb{Z}$-modules induces the zero map from $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$). On the other hand, suppose that $\varphi : M \to N$ is a surjective R-module homomorphism. The tensor product $D \otimes_R N$ is generated as an abelian group by the simple tensors $d \otimes n$ for $d \in D$ and $n \in N$. The surjectivity of φ implies that $n = \varphi(m)$ for some $m \in M$, and then $1 \otimes \varphi(d \otimes m) = d \otimes \varphi(m) = d \otimes n$ shows that $1 \otimes \varphi$ is a surjective homomorphism of abelian groups from $D \otimes_R M$ to $D \otimes_R N$. This proves most of the following theorem.

Theorem 39. Suppose that D is a right R-module and that L, M and N are left R-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence of abelian groups

$$D \otimes_R L \xrightarrow{1\otimes\psi} D \otimes_R M \xrightarrow{1\otimes\varphi} D \otimes_R N \longrightarrow 0 \quad \text{is exact.} \tag{10.13}$$

If D is an (S, R)-bimodule then (13) is an exact sequence of left S-modules. In particular, if $S = R$ is a commutative ring, then (13) is an exact sequence of R-modules with respect to the standard R-module structures. The map $1 \otimes \varphi$ is not in general injective, i.e., the sequence (13) cannot in general be extended to a short exact sequence.

The sequence (13) is exact for *all* right R-modules D if and only if

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \quad \text{is exact.}$$

Proof: For the first statement it remains to prove the exactness of (13) at $D \otimes_R M$. Since $\varphi \circ \psi = 0$, we have

$$(1 \otimes \varphi) \left(\sum d_i \otimes \psi(l_i)\right) = \sum d_i \otimes (\varphi \circ \psi(l_i)) = 0$$

and it follows that $\text{image}(1 \otimes \psi) \subseteq \ker(1 \otimes \varphi)$. In particular, there is a natural projection $\pi : (D \otimes_R M)/\ \text{image}(1 \otimes \psi) \to (D \otimes_R M)/\ker(1 \otimes \varphi) = D \otimes_R N$. The composite of the two projection homomorphisms

$$D \otimes_R M \to (D \otimes_R M)/\ \text{image}(1 \otimes \psi) \xrightarrow{\pi} D \otimes_R N$$

is the quotient of $D \otimes_R M$ by $\ker(1 \otimes \varphi)$, so is just the map $1 \otimes \varphi$. We shall show that π is an isomorphism, which will show that the kernel of $1 \otimes \varphi$ is just the kernel of the first projection above, i.e., $\text{image}(1 \otimes \psi)$, giving the exactness of (13) at $D \otimes_R M$. To see that π is an isomorphism we define an inverse map. First define $\pi' : D \times N \to (D \otimes_R M)/\ \text{image}(1 \otimes \psi)$ by $\pi'((d, n)) = d \otimes m$ for any $m \in M$ with $\varphi(m) = n$. Note that this is well defined: any other element $m' \in M$ mapping to n differs from m by an element in $\ker \varphi = \text{image}\, \psi$, i.e., $m' = m + \psi(l)$ for some $l \in L$, and $d \otimes \psi(l) \in \text{image}(1 \otimes \psi)$. It is easy to check that π' is a balanced map, so induces a homomorphism $\tilde{\pi} : D \times N \to (D \otimes_R M)/\ \text{image}(1 \otimes \psi)$ with $\tilde{\pi}(d \otimes n) = d \otimes m$. Then $\tilde{\pi} \circ \pi(d \otimes m) = \tilde{\pi}(d \otimes \varphi(m)) = d \otimes m$ shows that $\tilde{\pi} \circ \pi = 1$. Similarly, $\pi \circ \tilde{\pi} = 1$, so that π and $\tilde{\pi}$ are inverse isomorphisms, completing the proof that (13) is exact. Note also that the injectivity of ψ was not required for the proof.

Finally, suppose (13) is exact for every right R-module D. In general, $R \otimes_R X \cong X$ for any left R-module X (Example 1 following Corollary 9). Taking $D = R$ the exactness of the sequence $L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ follows.

By Theorem 39, the sequence

$$0 \longrightarrow D \otimes_R L \xrightarrow{1\otimes\psi} D \otimes_R M \xrightarrow{1\otimes\varphi} D \otimes_R N \longrightarrow 0$$

is not in general exact since $1 \otimes \psi$ need not be injective. If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a *split* short exact sequence, however, then since tensor products commute with direct sums by Theorem 17, it follows that

$$0 \longrightarrow D \otimes_R L \xrightarrow{1\otimes\psi} D \otimes_R M \xrightarrow{1\otimes\varphi} D \otimes_R N \longrightarrow 0$$

is also a split short exact sequence.

The following result relating to modules D having the property that (13) can always be extended to a short exact sequence is immediate from Theorem 39:

Proposition 40. Let A be a right R-module. Then the following are equivalent:

(1) For any left R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M \xrightarrow{1\otimes\varphi} A \otimes_R N \longrightarrow 0$$

is also a short exact sequence.

(2) For any left R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is an exact sequence of left R-modules (i.e., $\psi : L \to M$ is injective) then $0 \to A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M$ is an exact sequence of abelian groups (i.e., $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective).

Definition. A right R-module A is called *flat* if it satisfies either of the two equivalent conditions of Proposition 40.

For a fixed right R-module D, the first part of Theorem 39 is referred to by saying that the functor $D \otimes_R$ __ is *right exact*.

Corollary 41. If D is a right R-module, then the functor $D \otimes_R$ __ from the category of left R-modules to the category of abelian groups is right exact. If D is an (S, R)-bimodule (for example when $S = R$ is commutative and D is given the standard R-module structure), then $D \otimes_R$ __ is a right exact functor from the category of left R-modules to the category of left S-modules. The functor is exact if and only if D is a flat R-module.

We have already seen some flat modules:

Corollary 42. Free modules are flat; more generally, projective modules are flat.

Proof: To show that the free R-module F is flat it suffices to show that for any injective map $\psi : L \to M$ of R-modules L and M the induced map $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is also injective. Suppose first that $F \cong R^n$ is a finitely generated free R-module. In this case $F \otimes_R L = R^n \otimes_R L \cong L^n$ since $R \otimes_R L \cong L$ and tensor products commute with direct sums. Similarly $F \otimes_R M \cong M^n$ and under these isomorphisms the map $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is just the natural map of L^n to M^n induced by the inclusion ψ in each component. In particular, $1 \otimes \psi$ is injective and it follows that any finitely generated free module is flat. Suppose now that F is an arbitrary free module and that the element $\sum f_i \otimes l_i \in F \otimes_R L$ is mapped to 0 by $1 \otimes \psi$. This means that the element $\sum (f_i, \psi(l_i))$ can be written as a sum of generators as in equation (6) in the previous section in the free group on $F \times M$. Since this sum of elements is finite, all of the first coordinates of the resulting equation lie in some finitely generated free submodule F' of F. Then this equation implies that $\sum f_i \otimes l_i \in F' \otimes_R L$ is mapped to 0 in $F' \otimes_R M$. Since F' is a finitely generated free module, the injectivity we proved above shows that $\sum f_i \otimes l_i$ is 0 in $F' \otimes_R L$ and so also in $F \otimes_R L$. It follows that $1 \otimes \psi$ is injective and hence that F is flat.

Suppose now that P is a projective module. Then P is a direct summand of a free module F (Proposition 30), say $F = P \oplus P'$. If $\psi : L \to M$ is injective then $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is also injective by what we have already shown. Since $F = P \oplus P'$ and tensor products commute with direct sums, this shows that

$$1 \otimes \psi : (P \otimes_R L) \oplus (P' \otimes_R L) \to (P \otimes_R M) \oplus (P' \otimes_R M)$$

is injective. Hence $1 \otimes \psi : P \otimes_R L \to P \otimes_R M$ is injective, proving that P is flat.

Examples

(1) Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module it is flat. The example before Theorem 39 shows that $\mathbb{Z}/2\mathbb{Z}$ not a flat $\mathbb{Z}$-module.

(2) The $\mathbb{Z}$-module $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, as follows. Suppose $\psi : L \to M$ is an injective map of $\mathbb{Z}$-modules. Every element of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ can be written in the form $(1/d) \otimes l$ for some nonzero integer d and some $l \in L$ (Exercise 7 in Section 4). If $(1/d) \otimes l$ is in the kernel of $1 \otimes \psi$ then $(1/d) \otimes \psi(l)$ is 0 in $\mathbb{Q} \otimes_{\mathbb{Z}} M$. By Exercise 8 in Section 4 this means $c\psi(l) = 0$ in M for some nonzero integer c. Then $\psi(c \cdot l) = 0$, and the injectivity of ψ implies $c \cdot l = 0$ in L. But this implies that $(1/d) \otimes l = (1/cd) \otimes (c \cdot l) = 0$ in L, which shows that $1 \otimes \psi$ is injective.

(3) The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective (by Proposition 36), but is not flat: the injective map $\psi(z) = 2z$ from $\mathbb{Z}$ to $\mathbb{Z}$ does not remain injective after tensoring with $\mathbb{Q}/\mathbb{Z}$ ($1 \otimes \psi : \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}$ has the nonzero element $(\frac{1}{2} + \mathbb{Z}) \otimes 1$ in its kernel — identifying $\mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ this is the statement that multiplication by 2 has the element 1/2 in its kernel).

(4) The direct sum of flat modules is flat (Exercise 5). In particular, $\mathbb{Q} \oplus \mathbb{Z}$ is flat. This module is neither projective nor injective (since $\mathbb{Q}$ is not projective by Exercise 8 and $\mathbb{Z}$ is not injective by Proposition 36 (cf. Exercises 3 and 4).

We close this section with an important relation between Hom and tensor products:

Theorem 43. *(Adjoint Associativity)* Let R and S be rings, let A be a right R-module, let B be an (R, S)-bimodule and let C be a right S-module. Then there is an isomorphism of abelian groups:

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

(the homomorphism groups are right module homomorphisms—note that $\mathrm{Hom}_S(B, C)$ has the structure of a right R-module, cf. the exercises). If $R = S$ is commutative this is an isomorphism of R-modules with the standard R-module structures.

Proof: Suppose $\varphi : A \otimes_R B \to C$ is a homomorphism. For any fixed $a \in A$ define the map $\Phi(a)$ from B to C by $\Phi(a)(b) = \varphi(a \otimes b)$. It is easy to check that $\Phi(a)$ is a homomorphism of right S-modules and that the map Φ from A to $\mathrm{Hom}_S(B, C)$ given by mapping a to $\Phi(a)$ is a homomorphism of right R-modules. Then $f(\varphi) = \Phi$ defines a group homomorphism from $\mathrm{Hom}_S(A \otimes_R B, C)$ to $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. Conversely, suppose $\Phi : A \to \mathrm{Hom}_S(B, C)$ is a homomorphism. The map from $A \times B$ to C defined by mapping (a, b) to $\Phi(a)(c)$ is an R-balanced map, so induces a homomorphism φ from $A \otimes_R B$ to C. Then $g(\Phi) = \varphi$ defines a group homomorphism inverse to f and gives the isomorphism in the theorem.

As a first application of Theorem 43 we give an alternate proof of the first result in Theorem 39 that the tensor product is right exact in the case where $S = R$ is a commutative ring. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact, then by Theorem 33 the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(L, E)$$

is exact for every R-module E. Then by Theorem 28, the sequence

$$0 \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(N, E)) \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(M, E)) \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(L, E))$$

is exact for all D and all E. By adjoint associativity, this means the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D \otimes_R N, E) \longrightarrow \mathrm{Hom}_R(D \otimes_R M, E) \longrightarrow \mathrm{Hom}_R(D \otimes_R L, E)$$

is exact for any D and all E. Then, by the second part of Theorem 33, it follows that the sequence

$$D \otimes_R L \longrightarrow D \otimes_R M \longrightarrow D \otimes_R N \longrightarrow 0$$

is exact for all D, which is the right exactness of the tensor product.

As a second application of Theorem 43 we prove that the tensor product of two projective modules over a commutative ring R is again projective (see also Exercise 9 for a more direct proof).

Corollary 44. If R is commutative then the tensor product of two projective R-modules is projective.

Proof: Let P_1 and P_2 be projective modules. Then by Corollary 32, $\mathrm{Hom}_R(P_2, \_\_)$ is an exact functor from the category of R-modules to the category of R-modules. Then the composition $\mathrm{Hom}_R(P_1, \mathrm{Hom}_R(P_2, \_\_))$ is an exact functor by the same corollary. By Theorem 43 this means that $\mathrm{Hom}_R(P_1 \otimes_R P_2, \_\_)$ is an exact functor on R-modules. It follows again from Corollary 32 that $P_1 \otimes_R P_2$ is projective.

Summary

Each of the functors $\mathrm{Hom}_R(A, \_\_)$, $\mathrm{Hom}_R(\_\_, A)$, and $A \otimes_R \_\_$, map left R-modules to abelian groups; the functor $\_\_ \otimes_R A$ maps right R-modules to abelian groups. When R is commutative all four functors map R-modules to R-modules.

(1) Let A be a left R-module. The functor $\mathrm{Hom}_R(A, \_\_)$ is covariant and left exact; the module A is projective if and only if $\mathrm{Hom}_R(A, \_\_)$ is exact (i.e., is also right exact).

(2) Let A be a left R-module. The functor $\mathrm{Hom}_R(\_\_, A)$ is contravariant and left exact; the module A is injective if and only if $\mathrm{Hom}_R(\_\_, A)$ is exact.

(3) Let A be a right R-module. The functor $A \otimes_R \_\_$ is covariant and right exact; the module A is flat if and only if $A \otimes_R \_\_$ is exact (i.e., is also left exact).

(4) Let A be a left R-module. The functor $\_\_ \otimes_R A$ is covariant and right exact; the module A is flat if and only if $\_\_ \otimes_R A$ is exact.

(5) Projective modules are flat. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective but not flat. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is flat but neither projective nor injective.

EXERCISES

Let R be a ring with 1.

1. Suppose that

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' \end{array}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(a) if φ and α are surjective, and β is injective then γ is injective. [If $c \in \ker \gamma$, show there is a $b \in B$ with $\varphi(b) = c$. Show that $\varphi'(\beta(b)) = 0$ and deduce that $\beta(b) = \psi'(a')$ for some $a' \in A'$. Show there is an $a \in A$ with $\alpha(a) = a'$ and that $\beta(\psi(a)) = \beta(b)$. Conclude that $b = \psi(a)$ and hence $c = \varphi(b) = 0$.]

(b) if ψ', α, and γ are injective, then β is injective,

(c) if φ, α, and γ are surjective, then β is surjective,

(d) if β is injective, α and γ are surjective, then γ is injective,

(e) if β is surjective, γ and ψ' are injective, then α is surjective.

2. Suppose that

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

is a commutative diagram of groups, and that the rows are exact. Prove that

(a) if α is surjective, and β, δ are injective, then γ is injective.

(b) if δ is injective, and α, γ are surjective, then β is surjective.

3. Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective.

4. Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is an injective R-module if and only if both Q_1 and Q_2 are injective.

5. Let A_1 and A_2 be R-modules. Prove that $A_1 \oplus A_2$ is a flat R-module if and only if both A_1 and A_2 are flat. More generally, prove that an arbitrary direct sum $\sum A_i$ of R-modules is flat if and only if each A_i is flat. [Use the fact that tensor product commutes with arbitrary direct sums.]

6. Prove that the following are equivalent for a ring R:

(i) Every R-module is projective.

(ii) Every R-module is injective.

7. Let A be a nonzero finite abelian group.

(a) Prove that A is not a projective $\mathbb{Z}$-module.

(b) Prove that A is not an injective $\mathbb{Z}$-module.

8. Let Q be a nonzero divisible $\mathbb{Z}$-module. Prove that Q is not a projective $\mathbb{Z}$-module. Deduce that the rational numbers $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. [Show first that if F is any free module then $\cap_{n=1}^{\infty} nF = 0$ (use a basis of F to prove this). Now suppose to the contrary that Q is projective and derive a contradiction from Proposition 30(4).]

9. Assume R is commutative with 1.

(a) Prove that the tensor product of two free R-modules is free. [Use the fact that tensor products commute with direct sums.]

(b) Use (a) to prove that the tensor product of two projective R-modules is projective.

10. Let R and S be rings with 1 and let M and N be left R-modules. Assume also that M is an (R, S)-bimodule.

(a) For $s \in S$ and for $\varphi \in \mathrm{Hom}_R(M, N)$ define $(s\varphi) : M \to N$ by $(s\varphi)(m) = \varphi(ms)$. Prove that $s\varphi$ is a homomorphism of left R-modules, and that this action of S on $\mathrm{Hom}_R(M, N)$ makes it into a *left* S-module.

(b) Let $S = R$ and let $M = R$ (considered as an (R, R)-bimodule by left and right ring multiplication on itself). For each $n \in N$ define $\varphi_n : R \to N$ by $\varphi_n(r) = rn$, i.e., φ_n is the unique R-module homomorphism mapping 1_R to n. Show that $\varphi_n \in \mathrm{Hom}_R(R, N)$. Use part (a) to show that the map $n \mapsto \varphi_n$ is an isomorphism of left R-modules: $N \cong \mathrm{Hom}_R(R, N)$.

(c) Deduce that if N is a free (respectively, projective, injective, flat) left R-module, then $\mathrm{Hom}_R(R, N)$ is also a free (respectively, projective, injective, flat) left R-module.

11. Let R and S be rings with 1 and let M and N be left R-modules. Assume also that N is an (R, S)-bimodule.

(a) For $s \in S$ and for $\varphi \in \mathrm{Hom}_R(M, N)$ define $(\varphi s) : M \to N$ by $(\varphi s)(m) = \varphi(m)s$. Prove that φs is a homomorphism of left R-modules, and that this action of S on $\mathrm{Hom}_R(M, N)$ makes it into a *right* S-module. Deduce that $\mathrm{Hom}_R(M, R)$ is a right R-module, for any R-module M—called the *dual module* to M.

(b) Let $N = R$ be considered as an (R, R)-bimodule as usual. Under the action defined in part (a) show that the map $r \mapsto \varphi_r$ is an isomorphism of right R-modules: $\mathrm{Hom}_R(R, R) \cong R$, where φ_r is the homomorphism that maps 1_R to r. Deduce that if M is a finitely generated free left R-module, then $\mathrm{Hom}_R(M, R)$ is a free right R-module of the same rank. (cf. also Exercise 13.)

(c) Show that if M is a finitely generated projective R-module then its dual module $\mathrm{Hom}_R(M, R)$ is also projective.

12. Let A be an R-module, let I be any nonempty index set and for each $i \in I$ let B_i be an R-module. Prove the following isomorphisms of abelian groups; when R is commutative prove also that these are R-module isomorphisms. (Arbitrary direct sums and direct products of modules are introduced in Exercise 20 of Section 3.)

(a) $\mathrm{Hom}_R(\bigoplus_{i\in I} B_i, A) \cong \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$

(b) $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$.

13. **(a)** Show that the dual of the free $\mathbb{Z}$-module with countable basis is not free. [Use the preceding exercise and Exercise 24, Section 3.] (See also Exercise 5 in Section 11.3.)

(b) Show that the dual of the free $\mathbb{Z}$-module with countable basis is also not projective. [You may use the fact that any submodule of a free $\mathbb{Z}$-module is free.]

14. Let $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ be a sequence of R-modules.

(a) Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [To show the sequence splits, take $D = N$ and show the lift of the identity map in $\mathrm{Hom}_R(N, N)$ to $\mathrm{Hom}_R(N, M)$ is a splitting homomorphism for φ.]

(b) Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence.

15. Let M be a left R-module where R is a ring with 1.

(a) Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the action $(r\varphi)(r') = \varphi(r'r)$ (see Exercise 10).

(b) Suppose that $0 \to A \xrightarrow{\psi} B$ is an exact sequence of R-modules. Prove that if every homomorphism f from A to M lifts to a homomorphism F from B to M with $f = F \circ \psi$, then every homomorphism f' from A to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ lifts to a homomorphism F' from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ with $f' = F' \circ \psi$. [Given f', show that $f(a) = f'(a)(1_R)$ defines a homomorphism of A to M. If F is the associated lift of f to B, show that $F'(b)(r) = F(rb)$ defines a homomorphism from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ that lifts f'.]

(c) Prove that if Q is an injective R-module then $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is also an injective R-module.

16. This exercise proves Theorem 38 that every left R-module M is contained in an injective left R-module.

(a) Show that M is contained in an injective $\mathbb{Z}$-module Q. [M is a $\mathbb{Z}$-module—use Corollary 37.]

(b) Show that $\mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, Q)$.

(c) Use the R-module isomorphism $M \cong \mathrm{Hom}_R(R, M)$ (Exercise 10) and the previous exercise to conclude that M is contained in an injective module.

17. This exercise completes the proof of Proposition 34. Suppose that Q is an R-module with the property that every short exact sequence $0 \to Q \to M_1 \to N \to 0$ splits and suppose that the sequence $0 \to L \xrightarrow{\psi} M$ is exact. Prove that every R-module homomorphism f from L to Q can be lifted to an R-module homomorphism F from M to Q with $f = F \circ \psi$. [By the previous exercise, Q is contained in an injective R-module. Use the splitting property together with Exercise 4 (noting that Exercise 4 can be proved using (2) in Proposition 34 as the definition of an injective module).]

18. Prove that the injective hull of the $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathbb{Q}$. [Let H be the injective hull of $\mathbb{Z}$ and argue that $\mathbb{Q}$ contains an isomorphic copy of H. Use the divisibility of H to show $1/n \in H$ for all nonzero integers n, and deduce that $H = \mathbb{Q}$.]

19. If F is a field, prove that the injective hull of F is F.

20. Prove that the polynomial ring $R[x]$ in the indeterminate x over the commutative ring R is a flat R-module.

21. Let R and S be rings with 1 and suppose M is a right R-module, and N is an (R, S)-bimodule. If M is flat over R and N is flat as an S-module prove that $M \otimes_R N$ is flat as a right S-module.

22. Suppose that R is a commutative ring and that M and N are flat R-modules. Prove that $M \otimes_R N$ is a flat R-module. [Use the previous exercise.]

23. Prove that the (right) module $M \otimes_R S$ obtained by changing the base from the ring R to the ring S (by some homomorphism $f : R \to S$ with $f(1_R) = 1_S$, cf. Example 6 following Corollary 12 in Section 4) of the flat (right) R-module M is a flat S-module.

24. Prove that A is a flat R-module if and only if for any left R-modules L and M where L is *finitely generated*, then $\psi : L \to M$ injective implies that also $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective. [Use the techniques in the proof of Corollary 42.]

25. *(A Flatness Criterion)* Parts (a)-(c) of this exercise prove that A is a flat R-module if and only if for every finitely generated ideal I of R, the map from $A \otimes_R I \to A \otimes_R R \cong A$ induced by the inclusion $I \subseteq R$ is again injective (or, equivalently, $A \otimes_R I \cong AI \subseteq A$).

(a) Prove that if A is flat then $A \otimes_R I \to A \otimes_R R$ is injective.

(b) If $A \otimes_R I \to A \otimes_R R$ is injective for every finitely generated ideal I, prove that $A \otimes_R I \to A \otimes_R R$ is injective for every ideal I. Show that if K is any submodule of a finitely generated free module F then $A \otimes_R K \to A \otimes_R F$ is injective. Show that the same is true for any free module F. [Cf. the proof of Corollary 42.]

(c) Under the assumption in (b), suppose L and M are R-modules and $L \xrightarrow{\psi} M$ is injective. Prove that $A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M$ is injective and conclude that A is flat. [Write M as a quotient of the free module F, giving a short exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{f} M \longrightarrow 0.$$

Show that if $J = f^{-1}(\psi(L))$ and $\iota : J \to F$ is the natural injection, then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & J & \longrightarrow & L & \longrightarrow & 0 \\ & & {\scriptstyle id}\Big\downarrow & & {\scriptstyle \iota}\Big\downarrow & & {\scriptstyle \psi}\Big\downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative with exact rows. Show that the induced diagram

$$\begin{array}{ccccccc} A \otimes_R K & \longrightarrow & A \otimes_R J & \longrightarrow & A \otimes_R L & \longrightarrow & 0 \\ {\scriptstyle id}\Big\downarrow & & {\scriptstyle 1\otimes\iota}\Big\downarrow & & {\scriptstyle 1\otimes\psi}\Big\downarrow & & \\ A \otimes_R K & \longrightarrow & A \otimes_R F & \longrightarrow & A \otimes_R M & \longrightarrow & 0 \end{array}$$

is commutative with exact rows. Use (b) to show that $1 \otimes \iota$ is injective, then use Exercise 1 to conclude that $1 \otimes \psi$ is injective.]

(d) *(A Flatness Criterion for quotients)* Suppose $A = F/K$ where F is flat (e.g., if F is free) and K is an R-submodule of F. Prove that A is flat if and only if $FI \cap K = KI$ for every finitely generated ideal I of R. [Use (a) to prove $F \otimes_R I \cong FI$ and observe the image of $K \otimes_R I$ is KI; tensor the exact sequence $0 \to K \to F \to A \to 0$ with I to prove that $A \otimes_R I \cong FI/KI$, and apply the flatness criterion.]

26. Suppose R is a P.I.D. This exercise proves that A is a flat R-module if and only if A is torsion free R-module (i.e., if $a \in A$ is nonzero and $r \in R$, then $ra = 0$ implies $r = 0$).

(a) Suppose that A is flat and for fixed $r \in R$ consider the map $\psi_r : R \to R$ defined by multiplication by r: $\psi_r(x) = rx$. If r is nonzero show that ψ_r is an injection. Conclude from the flatness of A that the map from A to A defined by mapping a to ra is injective and that A is torsion free.

(b) Suppose that A is torsion free. If I is a nonzero ideal of R, then $I = rR$ for some nonzero $r \in R$. Show that the map ψ_r in (a) induces an isomorphism $R \cong I$ of

R-modules and that the composite $R \xrightarrow{\psi} I \xrightarrow{\iota} R$ of ψ_r with the inclusion $\iota : I \subseteq R$ is multiplication by r. Prove that the composite $A \otimes_R R \xrightarrow{1\otimes\psi_r} A \otimes_R I \xrightarrow{1\otimes\iota} A \otimes_R R$ corresponds to the map $a \mapsto ra$ under the identification $A \otimes_R R = A$ and that this composite is injective since A is torsion free. Show that $1 \otimes \psi_r$ is an isomorphism and deduce that $1 \otimes \iota$ is injective. Use the previous exercise to conclude that A is flat.

27. Let M, A and B be R-modules.

(a) Suppose $f : A \to M$ and $g : B \to M$ are R-module homomorphisms. Prove that $X = \{(a, b) \mid a \in A,\ b \in B \text{ with } f(a) = g(b)\}$ is an R-submodule of the direct sum $A \oplus B$ (called the *pullback* or *fiber product* of f and g) and that there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & M \end{array}$$

where π_1 and π_2 are the natural projections onto the first and second components.

(b) Suppose $f' : M \to A$ and $g' : M \to B$ are R-module homomorphisms. Prove that the quotient Y of $A \oplus B$ by $\{(f'(m), -g'(m)) \mid m \in M\}$ is an R-module (called the *pushout* or *fiber sum* of f' and g') and that there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \pi_2'} \\ A & \xrightarrow{\pi_1'} & Y \end{array}$$

where π_1' and π_2' are the natural maps to the quotient induced by the maps into the first and second components.

28. **(a)** *(Schanuel's Lemma)* If $0 \to K \to P \xrightarrow{\varphi} M \to 0$ and $0 \to K' \to P' \xrightarrow{\varphi'} M \to 0$ are exact sequences of R-modules where P and P' are projective, prove $P \oplus K' \cong P' \oplus K$ as R-modules. [Show that there is an exact sequence $0 \to \ker\pi \to X \xrightarrow{\pi} P \to 0$ with $\ker\pi \cong K'$, where X is the fiber product of φ and φ' as in the previous exercise. Deduce that $X \cong P \oplus K'$. Show similarly that $X \cong P' \oplus K$.]

(b) If $0 \to M \to Q \xrightarrow{\psi} L \to 0$ and $0 \to M \to Q' \xrightarrow{\psi'} L' \to 0$ are exact sequences of R-modules where Q and Q' are injective, prove $Q \oplus L' \cong Q' \oplus L$ as R-modules.

The R-modules M and N are said to be *projectively equivalent* if $M \oplus P \cong N \oplus P'$ for some projective modules P, P'. Similarly, M and N are *injectively equivalent* if $M \oplus Q \cong N \oplus Q'$ for some injective modules Q, Q'. The previous exercise shows K and K' are projectively equivalent and L and L' are injectively equivalent.