

Math 3593 Practice for the final exam.

You will **not** be allowed to use books, notes or a calculator on this exam. At the top of the exam you will be given the following formulas, as well as Taylor expansions of standard functions if you need them, but I don't think you do:

Formulas Surface area of a sphere: $4\pi r^2$. Volume of a sphere: $\frac{4}{3}\pi r^3$.

Polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$; $dx dy = r dr d\theta$.

Cylindrical coordinates: $x = r \cos \theta$, $y = r \sin \theta$, $z = z$; $dx dy dz = r dr d\theta dz$.

Spherical polars: $x = r \cos \phi \cos \theta$, $y = r \cos \phi \sin \theta$, $z = r \sin \phi$; $dx dy dz = r^2 \cos \phi dr d\phi d\theta$.

1. Prove that a subset of a set of volume zero has volume zero.
2. Find the surface area of the part of the graph of the function $z = y^2 - x^2$ which lies above the circle $x^2 + y^2 \leq 1$ in the xy -plane.
3. Let A be the unit circle $x^2 + y^2 \leq 1$ and let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation given by $\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - y \end{pmatrix}$. Find the area of $\Phi(A)$.
5. Find the length of the part of the helical spiral in \mathbb{R}^3 , specified in cylindrical polar coordinates (r, θ, z) by $r = 2$, $z = 3\theta$, for which $0 \leq \theta \leq 2\pi$.
6. Find the area of the plane elliptical region which is the part of the plane $z = 4 - x - 2y$ that lies above the circle $x^2 + y^2 \leq 1$ in the xy -plane.
7. Let ψ be the angle between a vector in \mathbb{R}^3 and the z -axis. Find the volume of the region in \mathbb{R}^3 bounded by the surface given in spherical polar coordinates by $r = 3(1 - \cos \psi)$.
8. Let $S = \partial B$ be the closed surface that is the boundary of the hemisphere

$$B : \quad x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.$$

Thus S is the union of the flat unit disc S_1 in the xy -plane given as

$$S_1 : \quad x^2 + y^2 \leq 1, \quad z = 0$$

and the curved surface S_2 given as

$$S_2 : \quad x^2 + y^2 + z^2 = 1, \quad z \geq 0.$$

Suppose that S is oriented with normal vector pointing out from the hemisphere at each point, and let S_1 and S_2 have this same orientation. Let F be the vector field $F(x, y, z) = (x + \cos y + \cos z, y + \sqrt{x^2 + 1} \ln(z^2 + 1), z + 3)$.

- (a) (4) Compute $\int_S F \cdot dS$.
- (b) (4) Compute $\int_{S_1} F \cdot dS$.
- (c) (4) Compute $\int_{S_2} F \cdot dS$.

9. Calculate

$$\int_{\gamma} (y - \tan^{-1} \sqrt{x+10}) dx + (3x + e^{y^2} \sin y) dy$$

where γ is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.

10. Let C be the curve in \mathbb{R}^2 parametrized by $\gamma(t) = \begin{pmatrix} t - t^2 \\ t - t^3 \end{pmatrix}$ where $0 \leq t \leq 1$, taken with the orientation given by this parametrization. You may assume that this curve is a loop which does not cross itself, and that it is in fact the boundary of a 2-manifold with boundary, namely the region enclosed by C .
- Calculate $\int_C y \, dx$.
 - By expressing the integral in (a) as a double integral (using Green's theorem), calculate the area of the region enclosed by C .
11. For each of the following sets, determine whether or not it is a smooth manifold, justifying your conclusion.
- The set of 2×2 real matrices A such that $A^2 = I$.
 - The set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ in \mathbb{R}^2 for which x and y have the same sign, or are both zero.
 - $\{x \in \mathbb{R} \mid x > 0\}$
 - $\mathbb{R} - \{0\}$
 - The union of the coordinate axes $x = 0$ and $y = 0$ in \mathbb{R}^2 .
12. Find the maximum and minimum values of $f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x^2 + xy + 2y^2 - z^2$ on the ball $x^2 + y^2 + z^2 \leq 100$.
- Plus: the questions which have appeared on the previous practice handouts, and
- Section 2.10: 2, 4, 7, 8, 14, 15, 16
 Section 2.11: 24, 29, 30, 31
 page 369 Section 3.7: no 6 Take this function and find its maximum and minimum values on $x^2 + y^2 + z^2 \leq 10$. (I have not done this modified problem and am not sure if it will work out appropriately.) Also questions 7, 8, 13, 16, 17 look relevant.
 p. 389 Section 3.9: 3.1, 3.2, 3.5, 3.10, 3.20
 p. 406: 4.1.10, 4.1.14
 p. 447: 4.5.7, 4.5.8, 4.5.11, 4.5.12, 4.5.14, 4.5.15, 4.5.16, 4.5.18, 4.5.19
 p. 476: 4.8.1, 4.8.2, 4.8.4, 4.8.13
 p. 495: 4.10.8, 4.10.12, 4.10.13, 4.10.14, 4.10.17, 4.10.18, 4.10.19
 p. 516 Section 4.12: 4.11, 4.12, 4.13, 4.21, 4.23
 p. 524: 5.1.1, 5.1.2
 p. 541: 5.3.2, 5.3.6, 5.3.8, 5.3.9, 5.3.15, 5.3.18, 5.3.21
 p. 556 Section 5.6: 5.3, 5.4