Math 8245 **Homework 4** **PJW**

Date due: November 1, 2010. Either hand it to me in class or put it in my mailbox by 3:30.

1. Let G be the free group on generators a and b. Prove that G is not generated by $\{a, bab\}$ or by $\{aba, bab\}$. [This shows that we cannot take the generating set $\{a, aba\}$ and replace a by either element of $\{b, bab\}$ and still have a generating set, in contrast to what happens with vector spaces. Generating sets do not form a matroid.]

2. Let F be free on generators x and y and let $\phi : F \to S_3$ be the homomorphism determined by $\phi(x) = (1,2)$ and $\phi(y) = (2,3)$. Let N be the kernel of ϕ. Find a set of free generators for N.
   [You may assume without proof that N is indeed a free group and that N is generated as a *normal* subgroup of F (not as a free group) by x^3, y^2 and $xyxy$.]

3. Let $G = \langle a, b \rangle$ where $a = (1,2,3,4)$ and $b = (3,4,5,6)$. Find a Schreier transversal in terms of a and b for $\mathrm{Stab}_G(2)$, the stabilizer in G of the symbol 2.

4. (Question 2 from page 114 of Johnson's book.) Prove in detail that in enumerating cosets for the presentation

$$T_n = \langle x, y \mid x^n y^{n+1}, x^{n+1} y^{n+2} \rangle, \quad n \in \mathbb{N},$$

   at least $n+1$ symbols are needed. Can you enlarge this lower bound?

5. Let `g` denote the group with presentation given in question 4. When $n = 3$ the command
   `gap> CosetTable(g,Subgroup(g,[]));`
   succeeds in showing that there is only one coset, but when $n = 20$ it fails.
   (i) Find the least value of n for which this command fails to enumerate the cosets, without increasing the default number of cosets allowed.
   (ii) Investigate what is going on and give an explanation, given that it is possible to show by coset enumeration that there is only one coset by introducing far fewer cosets than the default maximum. Does GAP simply make poor choices of cosets which it introduces?

6. The Mathieu group M_{12} may be generated by permutations

$$(1,2,3,4,5,6)(7,8,9,10,11,12) \quad \text{and} \quad (1,9,12,7,11)(6,2,8,3,5).$$

   a) Make a stabilizer chain for M_{12} and determine what the orbits $\Delta^{(i)}$ are.
   b) What is the smallest possible size of a base that a group of size $|M_{12}|$ acting on 12 points can have?
   c) For each i in $\{1, \ldots, 12\}$ find a word in the generators of M_{12} which sends 1 to i. [You may wish to modify the function `righttransversal` which formed part of

the code available in Lesson 6 so that it produces a word in the generators, or it is probably faster to do it by hand.]
d) The stabilizer of 1 in M_{12} is M_{11}. Compute a set of generators for $\mathrm{Stab}_{M_{12}}(1)$, expressing them as words in the generators of M_{12}.

7. Suppose that `grp` is a permutation group and that `sc:=StabChain(grp)` is a stabilizer chain for the group. The following is an attempt to write a function `iselement` of arguments `sc` and `g` which returns `true` precisely when `g` is a member of the permutation group `grp`.

```
iselement:=function(sc,g)
  local h,i;
  if sc.generators=[] then return g=();
  fi;
  h:=g;
  while sc.orbit[1]^h<> sc.orbit[1] do
    h:=h*sc.transversal[1^h];
  od;
  iselement(sc.stabilizer,h);
end;
```

There are some deliberate mistakes and omissions in the code. Make corrections to the code so that it works, correctly determining whether each of the permutations $(1,2,3,7,11)(4,9,6,10,5)$ and $(1,12)$ belong to

$$M_{11} = \langle (1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6) \rangle.$$

Answer the following questions:
(a) Why is the line `h:=g` present in the code? Is it necessary?
(b) Explain why it is that after some small changes of a typagrophical natrue the code will run without errors, but does not produce any answer. What should be done to correct this?

Extra questions – do not hand in:

8. Let F be a free group on a subset X. If $x \in X$ and $f \in F$, define $\sigma_x(f)$ to be the sum of the exponents of x in the reduced form of f. Prove that $f \in F'$ if and only if $\sigma_x(f) = 0$ for all x in X. [Here F' is the derived subgroup, which is generated by the commutators of elements in F.]

9. Write a function `OrbitInfo:=function(grp,i)` whose arguments are a permutation group `grp` and an integer `i`, which returns a list `[a,b]` where `a` is a list starting with `i` whose entries are the orbit containing `i` and where `b` is a list whose entries are either undefined or are taken from the given generators of `grp`, with the property that `b[j]` is defined if and only if `j` is in the same orbit as `i`, and then `i^b[j]` either appears earlier in `a` or is `a[1]`.
(You are thus asked to produce a list `[sc.orbit,sc.transversal]` where `sc` is a stabilizer chain. However, I would like you to write code for yourself, starting from scratch.)

10. The Mathieu group M_{24} may be generated by permutations

$$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14,15,16,17,18,19,20,21)(22,23)$$

and

$$(1,2,5,7,15,20,14,23,21,11,16,19,24,6,8,4,17,3,10,13,18)(9,22,12).$$

a) Make a stabilizer chain for M_{24} and determine the lengths of the orbits $\Delta^{(i)}$.
b) What is the smallest size of a base for a group of size $|M_{24}|$ acting on 24 points?
[I found the following function useful:

```
orbitlengths:=function(sc)
  local a;
  a:=ShallowCopy(sc);
  while IsBound(a.stabilizer) do
    Print("Orbit of length ", Length(a.orbit), "\n");
    a:=ShallowCopy(a.stabilizer);
  od;
  return;
end;
```

I used `ShallowCopy` a couple of times. Was it strictly necessary?]