Math 8246 **Homework 5** **PJW**
Date due: April 18, 2011.

1. Show that over any field $M^{(2,2)}$ has a series of submodules
$$0 \subset V_1 \subset V_2 \subset M^{(2,2)}$$
such that $V_1 \cong S^{(2,2)}$, $V_2/V_1 \cong S^{(3,1)}$, $M^{(2,2)}/V_2 \cong S^{(4)}$; and also a series of submodules
$$0 \subset W_1 \subset W_2 \subset M^{(2,2)}$$
such that $W_1 \cong S^{(1^4)}$, $W_2/W_1 \cong S^{(2,1,1)} \otimes S^{(1^4)}$ and $M^{(2,2)}/W_2 \cong S^{(2,2)} \otimes S^{(1^4)}$.
[Recall the aspects of structure done in class and treated in chapters 4 and 8. Recall question 2 on HW4, noting that $U \mapsto U^\circ$ is an order reversing bijection from the lattice of submodules of M to the lattice of submodules of M^*. Remember the information we have about dimensions of various modules. I have not gone over pages 20 and 21 of the text, but I think you can read what you need from these pages for yourself, and get suggestions for some useful mappings.]

2. For each of the following $(3,2)$-tabloids t express the polytabloid e_t as a linear combination of standard polytabloids:
$$t_a = \begin{matrix} 345 \\ 12 \end{matrix} \quad \text{and} \quad t_b = \begin{matrix} 543 \\ 21 \end{matrix}$$

3. (i) Find a basis for the Specht module $S^{(3,2)}$, each vector of which contains a unique standard tabloid in its support with respect to the basis of tabloids of the permutation module $M^{(3,2)}$. Express your basis for the Specht module as linear combinations of the standard polytabloids.
(ii) Find a basis for the subspace of $S^{(3,2)}$ consisting of linear combinations of tabloids which have no standard tabloid in their support.

4. When λ' is a p-regular partition, show that S^λ has a unique minimal submodule (which is isomorphic to $D^{\lambda'} \otimes S^{(1^n)}$).

5. (a) List all 2-regular particicitions of 4.
(b) Find the dimensions of all the irreducible representations of S_4 over $\mathbb{F}_2$.
(c) Working over $\mathbb{F}_2$, show that $S^{(2,1,1)}$ has a unique minimal submodule, and that in fact $S^{(2,1,1)}$ has a unique composition series. Identify the composition factors in this series.

6. (a) Working over $\mathbb{F}_3$, show that $S^{(2,2)}$ has two composition factors appearing in a unique composition series. Identify the isomorphism type of the unique minimal submodule of $S^{(2,2)}$.

(b) Working over $\mathbb{F}_3$, find the 3-regular particition λ such that $S^{(1^n)} \cong D^\lambda$.
(c) Find the decomposition matrix for S_4 in characteristic 3.

7. What is wrong with the following argument, which purports to show that in Theorem 9.3 the following stronger conclusion holds:

   INCORRECT THEOREM. Whenever μ is a partition of n and S^μ is regarded as an FS_{n-1}-module it is in fact a direct sum

$$S^\mu \downarrow_{S_{n-1}} \cong S^{\lambda^1} \oplus \cdots \oplus S^{\lambda^m}$$

   where $\lambda^1, \ldots \lambda^m$ are the partitions of $n-1$ which appear in the branching rule.

   Proof. In the argument given in the book, consider the short exact sequence of FS_{n-1}-modules $0 \to V_1 \to S^\mu \to S^\mu/V_1 \to 0$. The FS_{n-1}-module isomorphism $\theta_1 : V_1 \to S^{\lambda^1}$ is in fact defined on M^μ so we get a homomorphism $S^\mu \to S^{\lambda^1}$ which splits the short exact sequence. Therefore $S^\mu \cong S^{\lambda^1} \oplus S^\mu/V_1$. Now consider the short exact sequence $0 \to V_2/V_1 \to S^\mu/V_1 \to S^\mu/V_2 \to 0$. As before, the isomorphism $\theta_2 : V_2/V_1 \to S^{\lambda^2}$ extends to S^μ/V_1 (because it extends to M^μ/V_1) and so the sequence splits. Thus $S^\mu \cong S^{\lambda^1} \oplus S^{\lambda^2} \oplus S^\mu/V_2$. Continuing in this way we obtain that S^μ is a direct sum of the S^{λ^i}. □