1. Let r be prime and suppose that $xy \in (r) = \{rs \mid s \in R\}$. Then $xy = rs$ for some $s \in R$, so either r divides x or r divides y, so that one of x and y lies in (r). This means that (r) is a prime ideal.
Conversely, suppose that (r) is a prime ideal, and that r divides xy. This means $xy \in (r)$, so one of x, y lies in r, and is thus divisible by r.

2. We will suppose that R is an integral domain for this question. Let r be prime and suppose that $r = xy$. Then r divides xy, so r divides one of x or y because r is prime. If r divides x then $x = rs$ for some s, and $r = xy = rsy$. It follows that $sy = 1$ (because $r(1 - sy) = 0$ so $1 - sy = 0$ in an integral domain). Thus y is a unit. It is similar if r divides y. Thus r is irreducible.

3. Let R be a UFD and suppose that r is irreducible. Suppose that r divides xy, so $xy = rs$ for some s. Writing both sides in their unique irreducible factorizations, we see that the irreducible r must appear in the factorization of one of x or y, which means that r divides one of them, so r is prime.

4. Assuming that R satisfies (a) and (b), the missing condition we must show to see that R satisfies Eisenbud's definition of a UFD is that the expression in (b) is unique. For this, we will show by induction on n that if any element has a factorization $r = x_1 \cdots x_n$ as a product of irreducibles, then in any factorization $r = y_1 \cdots y_q$ into irrducibles we have $r = q$ and the irreducibles are equivalent after suitable permutation.
We start the induction with $n = 0$, which case r is a unit, and any factorization $r = y_1 \cdots y_q$ must be as a product of units, so the irreducibles in the factorization are the same (there aren't any). Assume $n > 0$ and the result is true for smaller values of n. Now x_1 divides $y_1 \cdots y_q$ and is prime, so x_1 divides some y_i. Thus $y_i = x_1 s$ for some s, which must be a unit because both x_1 and y_i are irreducible. It follows that $x_2 \cdots x_n = s y_1 \cdots \hat{y}_i \cdots y_q$, which are shorter products, so $n-1 = q-1$ and the factors are the same after permutation.

5. (i) Suupose that r is irreducible and $(r) \subseteq (s)$ for some element s where (s) is proper, so s is not a unit. Then $r = st$ for some element t, which must be a unit because r is irreducible. This means that $(r) = (s)$ so (r) is maximal.
Conversely, suppose (r) is maximal among proper principal ideals. Then if $r = st$ we have $(r) \subseteq (s)$ and $(r) \subseteq (t)$. We cannot have $(s) = (t) = R$ because then s, t are units, and r would be a unit, which it isn't. Thus if (s) is proper then $(r) = (s)$, which forces $s = ru$ for some u, and $r = st = rut$, so ut is a unit because R is a domain, so t is a unit. Thus r is irreducible.

5. (ii) For this part we should continue with the assumption that R is a domain. Suppose that R has ACC on principal ideals and consider an element $r \in R$. We show that r is a finite produce of irreducible elements. If not, then r is not irreducible and we can write $r = r1_1 r1_2$ where neither element is a unit and one of them is not irreducible. Write such a

non-irreducible element in the form $r2_ir2_j$, and repeat such factorizations. If we ever find that all our factorization conclude with irreducible elements, we will have factored r as a product of irreducible elements, so there exists an infinite chain $r1_{i_1}, r2_{i_2}, r3_{i_3}, \ldots$ where rj_{i_j} divides $r(j-1)_{i_{j-1}}$, so that $(r1_{i_1}) \subseteq (r2_{i_2}) \subseteq (r3_{i_3}) \subseteq \cdots$. This chain must terminate, so for some j, rj_{i_j} and $r(j-1)_{i_{j-1}}$ generate the same ideal, and because R is a domain they must differ by a unit, which contradicts the factorization they had. Thus 4(b) holds. For the implication 4(b) implies ACC on principal ideals an assumption is missing, and we should probably suppose R is a UFD. Sorry! Let $(r_1) \subseteq (r_2) \subseteq \cdots$ be an ascending chain of principal ideals. We get that each r_i is a factor of r_1, so its irreducible factors appear among the finitely many irreducible factors of r_1. Because there are only finitely many factors, the chain must terminate.

6. Eisenbud 1.1 on page 46.

1. $\Rightarrow$ 2. Suppose submodules of M are finitely generated and consider an ascending chain of submodules $M_1 \subseteq M_2 \subseteq \cdots$. Let $N = \bigcup M_i$. It is finitely generated, by elements $x_1, \ldots, x_d$. These elements must lie in some single term M_j and now the chain stabilizes with $M_j = M_{j+1} = \cdots$.
2.$\Rightarrow$ 3. Suppose ACC on submodules, and consider a set of submodules. If there is no maximal element in the set then for each submodule N_1 there is a larger submodule N_2, which has a larger submodule N_3, and by this means we construct an ascending chain that does not stabilize.
3.$\Rightarrow$ 4. The submodules $Rf_1 \subseteq Rf_1 + Rf_2 \subseteq Rf_1 + Rf_2 + Rf_3 \subseteq \cdots$ have a maximal element, so this chain terminates at some $Rf_1 + Rf_2 + \cdots + Rf_m$. Now if $n > m$ then f_n lies in this submodule, so can be written $\sum a_i f_i$.
4.$\Rightarrow$ 1. Let N be a submodule of M. If N is not finitely generated, having constructed $f_1, \ldots f_i$ we can find f_{i+1} in N and not in $Rf_1 + Rf_2 + \cdots + Rf_i$, and this constructs a sequence contravening condition 4.

7. Eisenbud 1.9 on page 49. Let X be an algebraic set. Under the correspondence, algebraic subsets of X correspond to radical ideals J with $J \supseteq I(X)$. If subsets X_1, X_2 correspond to radical ideals J_1, J_2 then $X_1 \cup X_2$ corresponds to J_1J_2. Thus $X = X_1 \cup X_2$ if and only if $J_1J_2 \subseteq I(X)$. If $I(X)$ is prime we cannot have $J_1J_2 \subseteq I(X)$ with ideals J_1J_2 that strictly contain $I(X)$, so X cannot be expressed as a union of proper algebraic subsets.
Conversely, if X cannot be expressed as a union of proper algebraic subsets, then $J_1J_2 \subseteq I(X)$ is not possible for radical ideals J_1, J_2 that strictly contain $I(X)$. For arbitrary ideals J_1, J_2 that strictly contain $I(X)$ we check that $J_1J_2 \subseteq I(X)$ implies that $(\text{rad } J_1)(\text{rad } J_2) \subseteq I(X)$, using the fact that $I(X)$ is radical, which again is not possible, so $I(X)$ is prime.

8. Eisenbud 1.24 on page 55, (a). We have that $\bigcap_j Z(I_j) = Z(\sum_j I_j$ without using the condition that the ideals in $Z(I)$ be prime, because an ideal contains all I_j if and only if it contains the ideal they generate, from the definition of the ideal they generate. To show

that finite unions of closed sets are closed we show that $Z(I_1) \cup \cdots \cup Z(I_n) = Z(I_1 \cdots I_n)$. To see this, every ideal in some $Z(I_j)$ contains $I_1 \cdots I_n$ because this ideal is contained in I_j, so the left side is contained in the right. Now suppose a prime ideal P lies in $Z(I_1 \cdots I_n)$, so it contains $I_1 \cdots I_n$. By iterating the prime condition we deduce that P contains some I_j, so lies in $Z(I_j)$. Thus the right side is contained in the left side, and they are equal.
(b) The closed subsets of $A^1(k)$ correspond to the maximal ideals of $k[x]$ that contain some ideal I of $k[x]$. Such an ideal has the form $I = (f)$ for some polynomial $f = r(x - a_1) \cdots (x - a_n)$ for some elements $a_1, \ldots, a_n, r \in k$, or $f = 0$. There are only finitely many elements $a_1, \ldots, a_n$ in such a closed set, unless $f = 0$, when we get the whole space. Thus the open sets are the complements of finite sets, and the empty set. Any two non-empty such open sets have non-empty intersection, so given to points v_1, v_2 it is not possible to find two disjoint open sets, each containing one of them. Thus the topology is not Hausdorff. In $A^2(k)$, the product of two proper open sets has infinite complement, so is not open in the Zariski topology. In the product topology it is open, so these two topologies are distinct.