Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 2 Due Thursday 10/21/2021, uploaded to Gradescope.

1. (2.10 of Eisenbud) Let R be a commutative ring. Show that every finitely generated module over $R[U^{-1}]$ is the localization of a finitely generated module over R. (Eisenbud also notes that the same implication without the condition finitely generated looks deeper but is a triviality. Do not address this comment.)

2. Let $A = M_{m,m}(R)$ and $B = M_{n,n}(R)$ be matrix rings over a commutative ring R. Show that $A \otimes_R B \cong M_{mn,mn}(R)$, where the multiplication giving the ring structure on the tensor product is determined by $(a \otimes b)(c \otimes d) := ac \otimes bd$ as in class and on page 65 of Eisenbud.

3. If R is any integral domain with quotient field Q, prove that

$$(Q/R) \otimes_R (Q/R) = 0.$$

4. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

5. (a) Let $K \supseteq \mathbb{Q}$ be a field containing $\mathbb{Q}$. Show that $K \otimes_{\mathbb{Q}} \mathbb{Q}[x] \cong K[x]$ as rings
(b) Show that, as a ring, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ is the direct sum of two fields.
[The ring multiplication is $(a \otimes b)(c \otimes d) := ac \otimes bd$ on basic tensors. Use the isomorphism $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2)$.]

6. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of R-modules, for some ring R. Suppose that A can be generated as an R-module by a subset $X \subseteq A$ and that C can be generated as an R-module by a subset $Y \subseteq C$. For each $y \in Y$, choose $y' \in B$ with $\beta(y') = y$. Prove that B is generated by the set $\alpha(X) \cup \{y' \mid y \in Y\}$.

7. Let A, B be left R-modules and let $r \in Z(R) = \{s \in R \mid st = ts \text{ for all } t \in R\}$, the center of R. Let $\mu_r : B \to B$ be multiplication by r. Prove that the induced map $(\mu_r)_* : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B)$ is also multiplication by r.

Extra questions: do not upload to Gradescope.

8. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

9. (Part of 2.4 of Eisenbud) Let k be a field and let m, n be integers. Describe as explicitly as possible the following. (For example, if the object is a finite-dimensional vector space, what is its dimension?)
a. $\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m))$
b. $k[x]/(x^n), \otimes_{k[x]} k[x]/(x^m)$

c. $k[x] \otimes_k k[x]$ (describe this as an algebra).

10. Eisenbud question 2.11 (It is very similar for modules to something we did for rings.

11. Let $U \subset R$ be a multiplicative subset not containing any zero divisors of a commutative ring R. We can regard R as the set of elements $\frac{r}{1}$ of $R[U^{-1}]$ where r ranges through R. If S is a ring with $R \subseteq S \subseteq R[U^{-1}]$, show that $S[U^{-1}] = R[U^{-1}]$.

12. Suppose that U and V are two multiplicative subsets of the commutative ring R with $U \subseteq V$. Writing V' for the image of V in $R[U^{-1}]$, show that $R[U^{-1}][V'^{-1}] = R[V^{-1}]$.