Chapter 2 of Eisenbud: Localization

2.1 Fractions

We learn:
What is a local ring?
How to invert 'multiplicative' sets of elements in a ring.

Big question: why would we want to do this?

Defn. A ring R is local $\iff$ it has a unique maximal ideal.

Examples. 1. Fields are local rings.

2. $\mathbb{Z}/4\mathbb{Z}$ has unique max ideal $2\mathbb{Z}/4\mathbb{Z}$

3. Let p be a prime

$$\cancel{\mathbb{Q}}_{(p)} = \left\{ \frac{a}{b} \;\middle|\; p \nmid b \right\} = \mathbb{Z}_{(p)}$$

Here (p) = ideal generated by $p = \frac{p}{1}$ in $\cancel{\mathbb{Q}}_{(p)}$ is the unique maximal ideal.

Pre-class Warm-up!

When p is a prime number, consider the subset of the rational numbers

$$\mathbb{Z}_{(p)} := \left\{ \tfrac{a}{b} \in \mathbb{Q} \,\middle|\, p \nmid b \right\}$$

What are the units in this ring?

A The elements $\frac{1}{b}$, $p \nmid b$

B The elements $\frac{a}{1}$, $a \neq 0$

✓C The elements $\frac{a}{b}$, $p \nmid a$, $p \nmid b$

D The elements $\frac{a}{b}$, $a \neq 0$, $p \nmid b$

E None of the above

Homework 1 is due on Thursday!

Proposition. An ideal I in a ring R is the unique maximal ideal if and only if every element of R - I is invertible.

Proof Suppose I is the unique maximal ideal of R.
If $r \in R - I$ and if $(r) \neq R$ then $(r) \subseteq I$ – contradiction. Thus $(r) = R$
$\exists\, s \in R$. $sr = 1$. Every element not in I is a unit.
Conversely, if every element not in I is a unit, and I is an ideal, if $J \subseteq R$ is an ideal then J consists of nonunits so $J \subseteq I$, I = unique maxl. ideal. $\square$

Corollary. $\mathbb{Z}_{(p)}$ is a local ring.

The units in $\mathbb{Z}_{(p)}$ are the set $\mathbb{Z}_{(p)} - p\mathbb{Z}_{(p)}$
b/c $p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \,\middle|\, \begin{matrix} p \nmid b \\ p \mid a \end{matrix} \right\}$
Thus $p\mathbb{Z}_{(p)}$ is the unique maxl. ideal of $\mathbb{Z}_{(p)}$.

The ideals of $\mathbb{Z}_{(p)}$ are
$\mathbb{Z}_{(p)} \supsetneq p\mathbb{Z}_{(p)} \supsetneq p^2\mathbb{Z}_{(p)} \supsetneq p^3\mathbb{Z}_{(p)}$.

The general form of localization = inverting some elements

In a ring R we start with a multiplicative set U:

Defn: If $u, v \in U$ then $u \cdot v \in U$ and $1 \in U$. We want elements $\frac{1}{u}$ in $R[U^{-1}]$

Eisenbud inverts U to get a ring R[U^-1] and also, given an R-module M he constructs a R[U^-1]-module M[U^-1]. He deduces the ring case from the module case. We can also do the ring case and then construct M[U^-1] as $R[U^{-1}] \otimes_R M$.

taking $M = R$

Problem 1. We haven't done $\otimes$ yet

2. $\otimes$ might not ~~be easy~~ to understand.

$M[U^{-1}]$ is an R-module, in fact, an $R[U^{-1}]$-module.

We want to write elements of $M[U^{-1}]$ as $\frac{m}{u}$ where $m \in M$, $u \in U$.

If $vm = 0$ for some $v \in U$. then $m = \frac{vm}{v} = 0$ in $M[U^{-1}]$

Also $\frac{m}{u} = \frac{m'}{u'} \iff \frac{mu' - m'u}{uu'} = 0$ which will happen if $\exists v \in U$, $v(mu' - m'u) = 0$

Formal definition of M[U^-1].

We define $\sim$ on $M \times U$

$(m,u) \sim (m',u') \iff \exists v \in U$ so that $v(mu' - m'u) = 0$

Define $M[U^{-1}]$ = the set of equivalence classes of $M \times U$ under $\sim$

Write the equivalence class of (m,u) as m/u.

$\underbrace{(m,u)} = \frac{m}{u}$. We define

$$\frac{m}{u} + \frac{m'}{u'} = \frac{mu' + m'u}{uu'}$$

Multiplicn by $r \in R$: $r\left(\frac{m}{u}\right) := \frac{rm}{u}$.

We have a map of R-modules M -> M[U^-1]

$$m \longmapsto \frac{m}{1}$$

When M = R we get a ring structure on R[U^-1] and M[U^-1] becomes a R[U^-1]-module.

Examples. 1. In $R[X]$ take the multiplicative subset $\underbrace{1, X, X^2, X^3, \ldots}_{= U}$

$R[X, U^{-1}] = R[X, X^{-1}]$

= ring of Laurent polynomials

$a_{-m}X^{-m} + a_{-m+1}X^{-m+1} + \cdots + a_n X^n$

$m, n \in \mathbb{Z}$.

2. $U = \mathbb{Z} - \{0\}$ $\quad \mathbb{Z}[U^{-1}] = \mathbb{Q}$

Question: what set of elements U did we invert to obtain $\mathbb{Z}_{(p)}$?

A {p}

B {p, p^2, p^3,…}

C The set of primes other than p.

D All integers not in the ideal (p) ✓

E None of the above.

Definition. If P is a prime ideal then $R-P$ is a multiplicative subset. $R[U^{-1}]$ where $U = R-P$ is also written R_P. e.g. if $P=(p) \subseteq \mathbb{Z}$ we write $\mathbb{Z}_{(p)}$.

The residue class field of R at P is
$\kappa = R[U^{-1}]/P \cdot R[U^{-1}]$

Note that $P \cdot R[U^{-1}]$ is the unique maximal ideal.

Question. For a prime integer p, what do you think the residue class field of $\mathbb{Z}$ at (p) is?

A $\mathbb{Z}_{(p)}$

B $\mathbb{Z}/p\mathbb{Z}$ ✓ The set of elements $\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots, \frac{p-1}{1}$ is a set of coset reps for $p\mathbb{Z}_{(p)}$ in $\mathbb{Z}_{(p)}$.

C $\mathbb{Q}$

D None of the above

Claim $\mathbb{Z}_{(p)}$ is $\mathbb{Z}[(\mathbb{Z}-(p))^{-1}]$

Universal property of the localization.

Why would we want to know about this?

Ideals of the localization.

Proposition 2.2. Ideals of R[U^-1] biject with ideals J of R for which the elements of U are non zero divisors on R/J.
Prime ideals of R[U^-1] biject with prime ideals of R not meeting U.

Example: The ideals of $\mathbb{Z}_{(p)}$.