Chapter 2 of Eisenbud: Localization

2.1 Fractions

We learn:
What is a local ring?
How to invert 'multiplicative' sets of elements in a ring.

Big question: why would we want to do this?

Defn. A ring R is local $\Leftrightarrow$ it has a unique maximal ideal.

Examples. 1. Fields are local rings.

2. $\mathbb{Z}/4\mathbb{Z}$ has unique max ideal $2\mathbb{Z}/4\mathbb{Z}$

3. Let p be a prime

$\cancel{\mathbb{Q}}_{(p)} = \{ \frac{a}{b} \mid p \nmid b \} = \mathbb{Z}_{(p)}$

Here (p) = ideal generated by $p = \frac{p}{1}$ in $\cancel{\mathbb{Q}}_{(p)}$ is the unique maximal ideal.

Pre-class Warm-up!

When p is a prime number, consider the subset of the rational numbers

$$\mathbb{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; p \nmid b \right\}$$

What are the units in this ring?

A The elements $\frac{1}{b}$, $p \nmid b$

B The elements $\frac{a}{1}$, $a \neq 0$

✓ C The elements $\frac{a}{b}$, $p \nmid a$, $p \nmid b$

D The elements $\frac{a}{b}$, $a \neq 0$, $p \nmid b$

E None of the above

Homework 1 is due on Thursday!

Proposition. An ideal I in a ring R is the unique maximal ideal if and only if every element of R - I is invertible.

Proof Suppose I is the unique maximal ideal of R. If $r \in R - I$ and if $(r) \neq R$ then $(r) \subseteq I$ – contradiction. Thus $(r) = R$ $\exists\, s \in R$ $sr = 1$. Every element not in I is a unit.

Conversely, if every element not in I is a unit, and I is an ideal, if $J \subseteq R$ is an ideal then J consists of nonunits so $J \subseteq I$, I = unique maxl. ideal. $\square$

Corollary. $\mathbb{Z}_{(p)}$ is a local ring.

The units in $\mathbb{Z}_{(p)}$ are the set $\mathbb{Z}_{(p)} - p\mathbb{Z}_{(p)}$ b/c $p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \,\middle|\, \begin{matrix} p \nmid b \\ p \mid a \end{matrix} \right\}$

Thus $p\mathbb{Z}_{(p)}$ is the unique maxl. ideal of $\mathbb{Z}_{(p)}$.

The ideals of $\mathbb{Z}_{(p)}$ are

$\mathbb{Z}_{(p)} \supseteq p\mathbb{Z}_{(p)} \supseteq p^2\mathbb{Z}_{(p)} \supseteq p^3\mathbb{Z}_{(p)}$.

The general form of localization = inverting some elements

In a ring R we start with a multiplicative set U:

Defn: If $u, v \in U$ then $u \cdot v \in U$ and $1 \in U$. We want elements $\frac{1}{u}$ in $R[U^{-1}]$

Eisenbud inverts U to get a ring R[U^-1] and also, given an R-module M he constructs a R[U^-1]-module M[U^-1]. He deduces the ring case from the module case. We can also do the ring case and then construct M[U^-1] as $R[U^{-1}] \otimes_R M$.

taking $M = R$

Problem 1. We haven't done $\otimes$ yet

2. $\otimes$ might not ~~be easy~~ to understand.

$M[U^{-1}]$ is an R-module, in fact, an $R[U^{-1}]$-module.

We want to write elements of $M[U^{-1}]$ as $\frac{m}{u}$ where $m \in M$, $u \in U$.

If $vm = 0$ for some $v \in U$.

then $m = \frac{vm}{v} = 0$ in $M[U^{-1}]$

Also $\frac{m}{u} = \frac{m'}{u'} \iff \frac{mu' - m'u}{uu'} = 0$ which will happen if $\exists v \in U$, $v(mu' - m'u) = 0$

Formal definition of M[U^-1].

We define $\sim$ on $M \times U$

$(m,u) \sim (m',u') \iff \exists v \in U$ so that $v(mu' - m'u) = 0$

Define $M[U^{-1}]$ = the set of equivalence classes of $M \times U$ under $\sim$

Write the equivalence class of (m,u) as m/u.

$\underbrace{(m,u)} = \frac{m}{u}$. We define

$$\frac{m}{u} + \frac{m'}{u'} = \frac{mu' + m'u}{uu'}$$

Multiplicn by $r \in R$: $r\left(\frac{m}{u}\right) := \frac{rm}{u}$.

We have a map of R-modules M -> M[U^-1]

$$m \longmapsto \frac{m}{1}$$

When M = R we get a ring structure on R[U^-1] and M[U^-1] becomes a R[U^-1]-module.

Examples. 1. In $R[X]$ take the multiplicative subset $\underbrace{1, X, X^2, X^3, \ldots}_{=U}$

$R[X, U^{-1}] = R[X, X^{-1}]$

= ring of Laurent polynomials

$a_{-m}X^{-m} + a_{-m+1}X^{-m+1} + \ldots + a_n X^n$

$m, n \in \mathbb{Z}$.

2. $U = \mathbb{Z} - \{0\}$ $\quad \mathbb{Z}[U^{-1}] = \mathbb{Q}$

Question: what set of elements U did we invert to obtain $\mathbb{Z}_{(p)}$?

A {p}

B {p, p^2, p^3,…}

C The set of primes other than p.

D All integers not in the ideal (p) ✓

E None of the above.

Pre-class Warm-up!!!!

Let k be the field of rational numbers and R = k[x,y] the polynomial ring in two variables.
What does the residue class field of R at the ideal (x) look like?

A k

B a finite degree extension of k (other than k)

C an infinite degree extension of k ✓

D a field of positive characteristic.

E none of the above

Discuss with someone else!!

Recall: (x) is a prime ideal.

$U = R - (x)$

$=$ polynomials not divis. by x.

$K = R[U^{-1}] / \underbrace{(x)R[U^{-1}]}_{\text{maximal ideal}}$.

$R[U^{-1}] = \left\{ \dfrac{f(x,y)}{g(x,y)} \;\middle|\; x \text{ is not a factor of } g \right\}$

$= R_{(x)}$

$\subseteq k(x,y)$

Factor out $(x)R[U^{-1}]$.

Put $x = 0$.

$K \cong \left\{ \dfrac{\hat{f}(y)}{\hat{g}(y)} \;\middle|\; \hat{f}, \hat{g} \in k[y],\ \hat{g} \neq 0 \right\}$

$= k(y)$

The elements $1, y, y^2, y^3, \ldots$ are independent over k.

Definition. If P is a prime ideal then $R-P$ is a multiplicative subset. $R[U^{-1}]$ where $U = R-P$ is also written R_P. e.g. if $P=(p) \subseteq \mathbb{Z}$ we write $\mathbb{Z}_{(p)}$.

The residue class field of R at P is $K = R[U^{-1}]/P\cdot R[U^{-1}]$

Note that $P\cdot R[U^{-1}]$ is the unique maximal ideal.

Question. For a prime integer p, what do you think the residue class field of $\mathbb{Z}$ at (p) is?

A $\mathbb{Z}_{(p)}$

B $\mathbb{Z}/p\mathbb{Z}$ ✓

C $\mathbb{Q}$

D None of the above

The set of elements $\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots, \frac{p-1}{1}$ is a set of coset reps for $p\mathbb{Z}_{(p)}$ in $\mathbb{Z}_{(p)}$.

Claim $\mathbb{Z}_{(p)}$ is $\mathbb{Z}[(\mathbb{Z}-(p))^{-1}]$

Conclude: b/c $\mathbb{Z}_{(p)}$ satisfies the universal property.

Universal property of the localization.

Given a ring hom. $\phi: R \to S$ so that $\forall u \in U$ $\phi(u)$ is a unit in S then $\exists$ unique map $\theta: R[U^{-1}] \to S$ so that the Δ commutes

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[dr] & S \\ & R[U^{-1}] \arrow[u, "\exists\theta"'] \end{tikzcd}$$

Proof. Define $\theta\left(\frac{r}{u}\right) := \phi(r)\phi(u)^{-1}$.

Why would we want to know about this?

Any ring with the same universal property as $R[U^{-1}]$ is isomorphic to $R[U^{-1}]$. (Given such a ring S we have

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \arrow[d, shift left] \\ & R[U^{-1}] \arrow[u, shift left] \end{tikzcd}$$

2 commut. Δs.

The two composites are the identity by uniqueness. so $S \cong R[U^{-1}]$.

Ideals of the localization.

Proposition 2.2. Ideals of R[U^-1] biject with ideals J of R for which the elements of U are non zero divisors on R/J.
Prime ideals of R[U^-1] biject with prime ideals of R not meeting U.

means: if $u \in U$ and $ur \in J$ then $r \in J$.

Example: The ideals of $\mathbb{Z}_{(p)}$.

are $\mathbb{Z}_{(p)} \supset p\mathbb{Z}_{(p)} \supseteq p^2\mathbb{Z}_{(p)} \cdots$

$U = \{ b \mid p \nmid b \}$.

Ideals of $\mathbb{Z}$ are (c), $c \in \mathbb{Z}$

When does U contain a zero divisor on $\mathbb{Z}/(c)$? It happens $\Leftrightarrow$ U contains a factor of c

We get such if $c = ap^n$

$a \neq 1$. $\gcd(a,p) = 1$.

Bijection is: ideals $(p^n) \subseteq \mathbb{Z} \leftrightarrow$ ideals of $\mathbb{Z}_{(p)}$

Proof. Let f : R -> R[U^-1] be the map r -> r/1.
We have maps

$$\begin{array}{ccc} \{\text{ideals of } R\} & \longleftrightarrow & \{\text{ideals of } R[U^{-1}]\} \\ J & \xrightarrow{\alpha} & f(J)R[U^{-1}] \\ f^{-1}(I) & \xleftarrow{\beta} & I \end{array}$$

preserve $\subseteq$

The composite $\alpha\beta$ is the identity.
Elements of I have form $\frac{r}{u}$, $u \in U$
so also $\frac{r}{1} = u \cdot \frac{r}{u} \in I$
$r \in f^{-1}(I)$, so $\frac{r}{u} \in f\, f^{-1}(I) \cdot R[U^{-1}]$.
Also $\alpha\beta(I) \subseteq I$.

The composite $\beta\alpha$ has $\beta\alpha(J) \supseteq J$ ✓

$\beta\alpha(J)$

We show that we have equality J = f^-1(f(J) R[U^-1] if and only if no element of U is a zero divisor on R/J.

Question: Is it
A easy, or B difficult to see that
Prime ideals of R[U^-1] biject with prime ideals of R not meeting U.

Proof. Let f : R -> R[U^-1] be the map r -> r/1.
We have maps

{ideals of R } <--> { ideals of R[U^-1] }

$$J \xrightarrow{\alpha} f(J)R[U^{-1}]$$

$$f^{-1}(I) \xleftarrow{\beta} I$$

The composite $\beta\alpha$ is the identity.
b/c

The composite $\alpha\beta$ has $\alpha\beta(J) \geq J$

We show that we have equality J = f^-1(f(J) R[U^-1]
if and only if no element of U is a zero divisor on
R/J.

"$\Rightarrow$" Suppose $J = \ldots$ and $ur \in J$ with $u \in U$, $r \in R$. We show $r \in J$.

Now $\frac{r}{1} = \frac{ur}{u} \in f(J)R[U^{-1}]$

so $r \in f^{-1}(\qquad) = J$. Done.

"$\Leftarrow$" Suppose no element of U is a zero divisor on R/J and let $r \in f^{-1}(f(J)R[U^{-1}])$. We show $r \in J$.

The supposition means $f(r) \in f(J)R[U^{-1}]$, so $\frac{r}{1} = \frac{s}{u}$ for some $s \in J$, $u \in U$

Thus $v(ur - s) = 0$ for some $v \in U$

$vur = vs \in J$. Thus $r \in J$ b/c vu is not a zero divisor on R/J.

$\square$

Question: On a scale 1: easy —- 10: difficult, how hard was that?

Corollary. A localization of a Noetherian ring is Noetherian.