Chapter 2 of Eisenbud: Localization

2.1 Fractions

We learn:
What is a local ring?
How to invert 'multiplicative' sets of elements in a ring.

Big question: why would we want to do this?

Defn. A ring R is local $\Leftrightarrow$ it has a unique maximal ideal.

Examples. 1. Fields are local rings.

2. $\mathbb{Z}/4\mathbb{Z}$ has unique max ideal $2\mathbb{Z}/4\mathbb{Z}$

3. Let p be a prime

$\mathbb{Q}_{(p)} = \left\{ \frac{a}{b} \mid p \nmid b \right\} = \mathbb{Z}_{(p)}$

Here (p) = ideal generated by $p = P$ in $\mathbb{Q}_{(p)}$ is the unique maximal ideal.

Pre-class Warm-up!

When p is a prime number, consider the subset of the rational numbers

$$\mathbb{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbb{Q} \,\middle|\, p \nmid b \right\}$$

What are the units in this ring?

A The elements $\frac{1}{b}$, $p \nmid b$

B The elements $\frac{a}{1}$, $a \neq 0$

✓C The elements $\frac{a}{b}$, $p \nmid a$, $p \nmid b$?

D The elements $\frac{a}{b}$, $a \neq 0$, $p \nmid b$

E None of the above

Homework 1 is due on Thursday!

Proposition. An ideal I in a ring R is the unique maximal ideal if and only if every element of R - I is invertible.

Proof Suppose I is the unique maximal ideal of R. If $r \in R - I$ and if $(r) \neq R$ then $(r) \subseteq I$ - contradiction. Thus $(r) = R$. $\exists\, s \in R$. $sr = 1$. Every element not in I is a unit.

Conversely, if every element not in I is a unit and I is an ideal if $J \subseteq R$ is an ideal then J consists of non-units so $J \subseteq I$, I = unique maxl. ideal. □

Corollary. $\mathbb{Z}_{(p)}$ is a local ring.

The units in $\mathbb{Z}_{(p)}$ are the set $\mathbb{Z}_{(p)} - p\mathbb{Z}_{(p)}$

b/c $p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \,\middle|\, \begin{matrix} p \nmid b \\ p \mid a \end{matrix} \right\}$

Thus $p\mathbb{Z}_{(p)}$ is the unique maxl. ideal of $\mathbb{Z}_{(p)}$.

The ideals of $\mathbb{Z}_{(p)}$ are

$\mathbb{Z}_{(p)} \supseteq p\mathbb{Z}_{(p)} \supseteq p^2\mathbb{Z}_{(p)} \supseteq p^3\mathbb{Z}_{(p)}$.

The general form of localization = inverting some elements

In a ring R we start with a multiplicative set U:

Defn: If $u, v \in U$ then $u \cdot v \in U$ and $1 \in U$. We want elements $\frac{f}{u}$ in $R[U^{-1}]$

Eisenbud inverts U to get a ring R[U^-1] and also, given an R-module M he constructs a R[U^-1]-module M[U^-1]. He deduces the ring case from the module case. We can also do the ring case and then construct M[U^-1] as $R[U^{-1}] \otimes_R M$.

taking $M = R$

Problem 1. We haven't done $\otimes$ yet

2. $\otimes$ might not be easy to understand.

$M[U^{-1}]$ is an R-module, in fact, an $R[U^{-1}]$-module.

We want to write elements of $M[U^{-1}]$ as $\frac{m}{u}$ where $m \in M$, $u \in U$.

If $vm = 0$ for some $v \in U$. then $m = \frac{vm}{v} = 0$ in $M[U^{-1}]$

Also $\frac{m}{u} = \frac{m'}{u'} \iff \frac{mu' - m'u}{uu'} = 0$ which will happen if $\exists v \in U$, $v(mu' - m'u) = 0$

Formal definition of M[U^-1].

We define $\sim$ on $M \times U$ $(m,u) \sim (m',u') \iff \exists v \in U$ so that $v(mu' - m'u) = 0$

Define $M[U^{-1}]$ = the set of equivalence classes of $M \times U$ under $\sim$

Write the equivalence class of (m,u) as m/u.

$(m,u) = \frac{m}{u}$. We define

$\frac{m}{u} + \frac{m'}{u'} = \frac{mu' + m'u}{uu'}$

Multiplich by $r \in R$: $r\left(\frac{m}{u}\right) := \frac{rm}{u}$.

We have a map of R-modules $M \to M[U^{-1}]$

$$m \longmapsto \frac{m}{1}$$

When $M = R$ we get a ring structure on $R[U^{-1}]$ and $M[U^{-1}]$ becomes a $R[U^{-1}]$-module.

Examples. 1. In $R[X]$ take the multiplicative subset $\underbrace{1, X, X^2, X^3, \ldots}_{= U}$

$R[X, U^{-1}] = R[X, X^{-1}]$

= ring of Laurent polynomials

$a_{-m} X^{-m} + a_{-m+1} X^{-m+1} + \cdots + a_n X^n$

$m, n \in \mathbb{Z}$.

2. $U = \mathbb{Z} - \{0\}$ $\quad \mathbb{Z}[U^{-1}] = \mathbb{Q}$

Question: what set of elements U did we invert to obtain $\mathbb{Z}_{(p)}$?

A {p}

B {p, p^2, p^3,…}

C The set of primes other than p.

D All integers not in the ideal (p) ✓

E None of the above.

Pre-class Warm-up!!!!

Let k be the field of rational numbers and R = k[x,y] the polynomial ring in two variables.
What does the residue class field of R at the ideal (x) look like?

A k

B a finite degree extension of k (other than k)

C an infinite degree extension of k ✓

D a field of positive characteristic.

E none of the above

Discuss with someone else!!

Recall: (x) is a prime ideal.

$U = R - (x)$

$=$ polynomials not divis. by x.

$K = R[U^{-1}] / (x)R[U^{-1}]$ maximal ideal.

$R[U^{-1}] = \left\{ \frac{f(x,y)}{g(x,y)} \mid x \text{ is not a factor of } g \right\}$

$\| R_{(x)}$

$\subseteq k(x,y)$

Factor out $(x)R[U^{-1}]$.

Put $x = 0$.

$k \cong \left\{ \frac{\hat{f}(y)}{\hat{g}(y)} \mid \hat{f}, \hat{g} \in k[y], \hat{g} \neq 0 \right\}$

$= k(y)$

The elements $1, y, y^2, y^3, \dots$ are independent over k.

Definition. If P is a prime ideal then $R-P$ is a multiplicative subset $R[U^{-1}]$ where $U = R-P$ is also written R_P. e.g. if $P=(p) \subseteq \mathbb{Z}$ we write $\mathbb{Z}_{(p)}$.

The residue class field of R at P is
$\kappa = R[U^{-1}]/P\cdot R[U^{-1}]$

Note that $P\cdot R[U^{-1}]$ is the unique maximal ideal.

Question. For a prime integer p, what do you think the residue class field of $\mathbb{Z}$ at (p) is?

A $\mathbb{Z}_{(p)}$

B $\mathbb{Z}/p\mathbb{Z}$ ✓ The set of elements $\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \dots, \frac{p-1}{1}$ is a set of coset reps for $p\mathbb{Z}_{(p)}$ in $\mathbb{Z}_{(p)}$.

C $\mathbb{Q}$

D None of the above

Claim $\mathbb{Z}_{(p)}$ IS $\mathbb{Z}[(\mathbb{Z}-(p))^{-1}]$

Conclude: b/c $\mathbb{Z}_{(p)}$ satisfies the universal property.

Universal property of the localization.

Given a ring hom. $\phi: R \to S$ so that $\forall u \in U$ $\phi(u)$ is a unit in S then $\exists$ unique map $\theta: R[U^{-1}] \to S$ so that the $\triangle$ commutes

Proof. Define $\theta(\frac{r}{u}) := \phi(r)\,\phi(u)^{-1}$.

Why would we want to know about this?

Any rng with the same universal property as $R[U^{-1}]$ is isomorphic to $R[U^{-1}]$.

(Given such a rng S we have

2 commut. $\triangle$s.

The two compositions are the identity by uniqueness.

so $S \cong R[U^{-1}]$.

Pre-class Warm-up!

What do you think $(\mathbb{Z}/6\mathbb{Z})_{(3)}$ is?,

namely, the localization of $\mathbb{Z}/6\mathbb{Z}$ at the prime p = 3?

A $\mathbb{Z}/6\mathbb{Z}$

B $\mathbb{Z}/2\mathbb{Z}$

C $\mathbb{Z}/3\mathbb{Z}$ ✓ 3

D $\mathbb{Z}_{(3)}$ 3

E None of the above. ∞

Second question: Can you prove it?

We could show $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ is universal among homs $\to S$ where 1,2,4,5 are invertible in S.

Intended: $\mathbb{Z}/6\mathbb{Z}$ is a $\mathbb{Z}$-module
$(3) \subseteq \mathbb{Z}$ $(\mathbb{Z}/6\mathbb{Z})_{(3)}$ is a $\mathbb{Z}_{(3)}$-module

Other interpretation.

$\mathbb{Z}/6\mathbb{Z}$ is a ring (3) is a prime ideal of $\mathbb{Z}/6\mathbb{Z}$.

$\mathbb{Z}/6\mathbb{Z}$ is a $\mathbb{Z}$-module generated by a single element.

$(\mathbb{Z}/6\mathbb{Z})_{(3)}$ is $\mathbb{Z}_{(3)}$-module generated by a single element so has the form $\mathbb{Z}_{(3)}/I$ for some ideal. which must be $\mathbb{Z}_{(3)}$ or $3\mathbb{Z}_{(3)}$ or $9\mathbb{Z}_{(3)}$ or ... Giving 0 or $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$..

$3 \cdot (\mathbb{Z}/6\mathbb{Z})_{(3)} = 0$ b/c it is killed by 2, and $\frac{1}{2} \in \mathbb{Z}_{(3)}$.

$\frac{1}{1} = 0 \iff \exists\, u \in U$ with $u(\bar{1}-\bar{0}) = \bar{0}$

There is no such u prime to 3 so $\frac{1}{1} \neq 0$. Thus $\mathbb{Z}/6\mathbb{Z}_{(3)} \tilde{=} \mathbb{Z}/3\mathbb{Z}$

Ideals of the localization.

Proposition 2.2. Ideals of R[U^-1] biject with ideals J of R for which the elements of U are non zero divisors on R/J.
Prime ideals of R[U^-1] bijects with prime ideals of R not meeting U.

means: if $u \in U$ and $ur \in J$ then $r \in J$.

Example: The ideals of $\mathbb{Z}_{(p)}$.

are $\mathbb{Z}_{(p)} \supset p\mathbb{Z}_{(p)} \supset p^2\mathbb{Z}_{(p)} \cdots$

$U = \{ b \mid p \nmid b \}$.

Ideals of $\mathbb{Z}$ are (c), $c \in \mathbb{Z}$

When does U contain a zero divisor on $\mathbb{Z}/(c)$? It happens $\Leftrightarrow$ U contains a factor of c.

We get such if $c = ap^n$

$a \neq 1$. $\gcd(a,p) = 1$.

Bijection is: ideals $(p^n) \subseteq \mathbb{Z} \leftrightarrow$ ideals of $\mathbb{Z}_{(p)}$

Proof. Let $f : R \to R[U^{-1}]$ be the map $r \to r/1$.
We have maps

{ideals of R} <--> { ideals of $R[U^{-1}]$ }

$J \xrightarrow{\alpha} f(J)R[U^{-1}]$ preserve $\subseteq$

$f^{-1}(I) \xleftarrow{\beta} I$

The composite $\alpha\beta$ is the identity.
Elements of I have form $\frac{r}{u}$, $u \in U$
so also $f = u \cdot \frac{r}{u} \in I$
$r \in f^{-1}(I)$, so $\frac{r}{u} \in f f^{-1}(I) \cdot R[U^{-1}]$.
Also $\alpha\beta(I) \subseteq I$.

The composite $\beta\alpha$ has $\beta\alpha(J) \supseteq J$ ✓ $\beta\alpha(J) = J$

We show that we have equality $J = f^{-1}(f(J)R[U^{-1}])$ if and only if no element of U is a zero divisor on R/J.

Question: Is it
A easy, or B difficult to see that
Prime ideals of R[U^-1] bijject with prime ideals of R not meeting U.

Proof. Let f : R -> R[U^-1] be the map r -> r/1. We have maps

{ideals of R} <--> { ideals of R[U^-1] }

$J \xrightarrow{\alpha} f(J)R[U^{-1}]$

$f^{-1}(I) \xleftarrow{\beta} I$

The composite $\alpha\beta$ is the identity. b/c

The composite $\beta\alpha$ has $\beta\alpha(J) \supseteq J$

We show that we have equality $J = f^{-1}(f(J)\,R[U^{-1}])$ if and only if no element of U is a zero divisor on R/J.

"$\Rightarrow$" Suppose $J = \ldots$ and $ur \in J$ with $u \in U$, $r \in R$. We show $r \in J$.

Now $\frac{r}{1} = \frac{ur}{u} \in f(J)R[U^{-1}]$

so $r \in f^{-1}(\quad) = J$. Done.

Eisenbud seems not to do this.

"$\Leftarrow$" Suppose no element of U is zero divisor on R/J and let $r \in f^{-1}(f(J)R[U^{-1}])$. We show $r \in J$.

The supposition means $f(r) \in f(J)\,R[U^{-1}]$, so $\frac{r}{1} = \frac{s}{u}$ for some $s \in J$, $u \in U$

Thus $v(ur - s) = 0$ for some $v \in U$

$vur = vs \in J$. Thus $r \in J$ b/c vu is not a zero divisor on R/J.

1-1

Prime ideals? f induces a ring homom. $R/f^{-1}(I) \longrightarrow R[U^{-1}]/I$ so I prime $\Rightarrow f^{-1}(I)$ prime, in which case U nzd on $R/f^{-1}(I) \Leftrightarrow U \cap f^{-1}(I) = \emptyset$

Conversely, $f^{-1}(I)$ prime and $\frac{r}{u}\frac{s}{v} \in I$

$\Rightarrow \frac{rs}{1} \in I$, $rs \in f^{-1}(I)$ so $r \in f^{-1}I$ or $s \in f^{-1}I$ and $\frac{r}{u} \in I$ or $\frac{s}{v} \in I$. so I is prime.

Question: On a scale 1: easy —- 10: difficult, how hard was that?

Corollary. A localization of a Noetherian ring is Noetherian.

Proof: The ideals of $R[U^{-1}]$ bijject with a subset of ideals of R.

If ideals of R satisfy ACC so do the ideals of $R[U^{-1}]$ $\square$