Tensor products

In section 2.2 of his book, Eisenbud only does tensor products over a commutative ring, adopting a definition that does not apply in generality, so we have to follow a different source.

In my notes I denote a noncommutative ring by R, and it would be more suitable if the ring were called A, for instance.

We learn (for starters):

The definition of the tensor product of a right R-module M with a left R-module N. $M \otimes_R N$
What a tensor is.
What the tensor product looks like in special cases.
The tensor product of two matrices.
The adjoint property of Hom and Tensor Product.

$(0+0)\otimes n = 0\otimes n = 0\otimes n + 0 \otimes n$

$0 = 0 \otimes n$

Definition 2.1.1 (my notes) Given a right R-module M, a left R-module N, we construct the free abelian group T with basis (the elements of) $M\times N$

We construct the subgroup S generated by elements

$(m_1+m_2, n) - (m_1, n) - (m_2, n)$
$(m, n_1+n_2) - (m, n_1) - (m, n_2)$
$(mr, n) - (m, rn) \quad \forall\, m \in M, n \in N, r \in R$

Defn. $M \otimes_R N = T/S$

Basic tensors. Defn $m \otimes_R n := (m,n)+S$. An element of $M\otimes_R N$ is a tensor. An arbitrary tensor is a $\mathbb{Z}$-linear combination of basic tensors

Distributivity etc of tensors. The following hold:

$(m_1+m_2)\otimes_R n = m_1 \otimes_R n + m_2 \otimes_R n$
$m \otimes (n_1+n_2) = m\otimes_R n_1 + m \otimes_R n_2$
$mr \otimes n = m \otimes rn$ always.

Question: how difficult is it to see that $m\otimes 0 = 0 = 0 \otimes n$ always?

Scale Easy 1 2 ... 10 difficult

Definition. M is a right R-module, N a left R-module, L an abelian group.
A mapping f : M x N -> L is R-balanced if and only if

$$f(m_1+m_2,n) = f(m_1,n)+f(m_2,n)$$

$$f(m,n_1+n_2) = f(m,n_1)+f(m,n_2)$$

$$f(mr,n) = f(m,rn)$$

Example. We have a mapping

$$f: M\times N \to M\otimes_R N$$

$$f(m,n) = m\otimes_R n$$

If R is commutative an R-bilinear map $g: M\times N \to L$ an R-module has $g(r_1m_1+r_2m_2,n) = r_1g(m_1,n)+r_2g(m_2,n)$

$g(m, r_1n_1+r_2n_2) =$ similar.

Such g is balanced.

Discussion. What is the difference between the notion of being balanced and some concept of R-bilinear?

Is it a problem that f(rm,s~~u~~n) = rf(m,sn) = rsf(m,n) = srf(m,n)?

Yes

No

Theorem (Dummit and Foote Cor.11)
The balanced map $M \times N \to M \otimes_R N$ is universal with respect to balanced maps.
The tensor product $M \otimes_R N$ is defined up to isomorphism by this property.

Given a balanced map β

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[dr, "f"'] & L \\ & M \otimes_R N \arrow[u, "\alpha"'] \end{tikzcd}$$

$\exists$ unique homom of ab. gps $M \otimes_R N \xrightarrow{\alpha} L$

so that the $\triangle$ commutes.

Theorem. (D & F Theorem 10).
Balanced maps $M \times N \to L$ biject with group homomorphisms $M \otimes_R N \to L$.

Proof. Given a group homom $M \otimes_R N \to L$ we get a balanced map

$$M \times N \xrightarrow{f} M \otimes_R N \longrightarrow L$$

Given a balanced map

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr] & L \\ & M \otimes_R N \arrow[u, "\alpha"'] \end{tikzcd}$$

$\exists$ unique homom

The two constructions are inverse. $\square$

Pre-class Warm-up!!

True or false?:

The mapping $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \mathbb{Q}\oplus\mathbb{Q}$

specified by $a\otimes_{\mathbb{Z}} b \longmapsto (a,b)$

is a group homomorphism.

A True

B False ✓

$$a\otimes_{\mathbb{Z}} b + c\otimes_{\mathbb{Z}} d \longmapsto (a,b)+(c,d)$$
$$= (a+c, b+d) \overset{?}{=}$$

$$0\otimes 0 \longmapsto (0,0)$$
$$\shortparallel \qquad\qquad \neq$$
$$10\otimes 0 \longmapsto (10,0)$$

How can construct well-defined homomorphisms?

Is the assignment

$$\mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}\oplus\mathbb{Q}$$
$$(a,b) \to (a,b) \qquad \mathbb{Z}\text{-balanced ?}$$

$$? \; (ar,b) \to (ar,b)$$
$$\neq$$
$$(a,rb) \to (a,rb)$$

go to the same place.

Not $\mathbb{Z}$-balanced, don't get a homom.

Examples.

1. If f : R -> S is a ring homomorphism with f(1_)R) = 1_S then $S \otimes_R R \cong S$ as left S-modules.

Example: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$

We regard S as a right R-module via f

True or false?: $S \otimes_R R \cong S$

Proof that

The mapping

Defined a mapping $g : S \times R \to S$

$(s,r) \longmapsto s f(r)$

specified by

Claim: this is balanced!

is a group homomorphism.

$g(s_1+s_2, r) = g(s_1,r)+g(s_2,r)$

A True

$g(s, r_1+r_2) = g(s,r_1)+g(s,r_2)$

B False

$g(st, r) = g(s,tr)$ when $t \in R$.

i.e. $g(sf(t), r) = g(s,tr) = s f(tr)$

$= s f(t) f(r)$

How is $S \otimes_R R$ a left S-module?

Defn. $s_1(s \otimes r) := s_1 s \otimes r$. This is well defined because $S \times R \to S \otimes_R R$

Let M be an R-module.

2. Let I be a right ideal of R. Then

$(I \backslash R) \otimes_R M \cong M/IM$

Proof Construct isomorphisms (well-defined)

$1+I \otimes m \longleftarrow m + IM$

$(r+I) \otimes m \longrightarrow rm + IM$

using $I \backslash R \times M \longrightarrow M/IM$ is balanced

$(r+I, m) \longrightarrow rm + IM$

These are inverse, get an isomorphism

3. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

We have $S \otimes_R R \to S$. We also construct $S \otimes_R R \leftarrow S$: h a map of S-modules

$s \otimes 1_R \longmapsfrom s$

Check $hg = 1_{S \otimes_R R}$ $gh = 1_R$

Conclude $S \otimes_R R \cong S$.

$(s,r) \longrightarrow s_1 s \otimes_R r$ is balanced. (s,tr)

e.g. $(st, r) \to s_1 s f(t) \otimes_R r = s_1 s \otimes_R tr$

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

<u>Proof</u>. $m\mathbb{Z}$ is an ideal in $\mathbb{Z}$

so it is $(\mathbb{Z}/n\mathbb{Z}) / m\mathbb{Z}(\mathbb{Z}/n\mathbb{Z})$

$$\cong \mathbb{Z}/(n\mathbb{Z}+m\mathbb{Z})$$

$$= \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Pre-class Warm-up!

True or false?

The mapping $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \to \mathbb{Q}$

specified by $a\otimes_{\mathbb{Z}} b \mapsto ab$

is a group homomorphism.

A True ✓

B False.

There is also a map

$$1\otimes_{\mathbb{Z}} q \leftarrow q$$

$$1\otimes q + 1\otimes q = 1\otimes(q+q)$$
$$= 1\otimes 2q$$

The composite $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \leftrightarrows \mathbb{Q}$

is the identity.

What about the composite

$\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \leftrightarrows \mathbb{Q}$ is

$$\frac{r}{s}\otimes\frac{u}{v} \to 1\otimes_{\mathbb{Z}}\frac{ru}{sv}.$$

$$= r\otimes\frac{u}{sv}$$

$$s\left(1\otimes\frac{ru}{sv}\right) = 1\otimes\frac{ru}{v} = r\otimes\frac{u}{v}$$

Deduce that $1\otimes\frac{ru}{sv} = \frac{1}{s}\left(r\otimes\frac{u}{v}\right)$

$$= \frac{r}{s}\otimes\frac{u}{v}.$$

Thus $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathbb{Q}$

Theorem 1.8.
Tensor product distributes over direct sums:

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$

and similarly on the other side.

Proof. Define $\longrightarrow$ and $\longleftarrow$ that are inverse.

$(M \oplus M') \times N \to$ stuff on right.

$((m, m'), n) \to (m \otimes n, m' \otimes n)$

It's balanced

$(m, 0) \otimes n + (0, m') \otimes n' \longleftarrow (m \otimes n, m' \otimes n')$

check balanced for these

Question: Should this be easy or difficult to prove?

Easy 1 . . . 10 Difficult.

Question: Does this seem a reasonable proof?

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$.

Tensor products of vector spaces

Example 1.10 Let U and V be vector spaces over a field K with bases $u_1, \dots, u_m$, $v_1, \dots, v_n$.

Then $U \otimes_K V$ is a vector space with basis $u_i \otimes v_j$ $\quad 1 \le i \le m$, $1 \le j \le n$.

so $\dim(U \otimes_K V) = \dim U \cdot \dim V$.

This is because

$$U \otimes V = \left(\bigoplus K u_i\right) \otimes \left(\bigoplus K v_j\right)$$

$$= \bigoplus_{i,j} \left(K u_i \otimes_K K v_j\right)$$

$$= \bigoplus_{i,j} K(u_i \otimes v_j)$$

Definition 1.11. Let f : M -> M' and g : N -> N' be homomorphisms of right and left R-modules. We define

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$$

to be the group homomorphism determined by the balanced map $M \times N \longrightarrow M' \otimes_R N'$ given by $(m,n) \longmapsto f(m) \otimes g(n)$

$(f \otimes g)(m \otimes n) := f(m) \otimes g(n)$

Example 1.12

Check $(mr, n) \longmapsto f(mr) \otimes g n$

$= f(m) \cdot r \otimes g(n)$

$= f(m) \otimes r g(n)$

$= f(m) \otimes g(rn)$

(m, rn)

Let $R = K$ be a field $M = K^2 = N$

$f : K^2 \to K^2$ has matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

$g : K^2 \to K^2$ $\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix}$

Then we get $f \otimes g : K^2 \otimes K^2 \to K^2 \otimes K^2$

has matrix

$$\begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} & 0\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} \\ 2\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} & 3\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 10 & 0 & 15 \\ 8 & 12 & 12 & 18 \end{bmatrix}$$

This is the tensor product of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix}$.

Suppose the basis of M ~~are~~ is u_1, u_2 and of N is v_1, v_2.

A $u_1 \otimes v_1$, $u_2 \otimes v_2$, $u_1 \otimes v_2$, $u_2 \otimes v_1$

B ✓ $u_1 \otimes v_1$, $u_1 \otimes v_2$, $u_2 \otimes v_1$, $u_2 \otimes v_2$

C $u_1 \otimes v_1$, $u_2 \otimes v_1$, $u_1 \otimes v_2$, $u_2 \otimes v_2$

D None of the above!

$u_1 \otimes v_1 \longmapsto (u_1 + 2u_2) \otimes 4v_2$

$= 4u_1 \otimes v_2 + 8u_2 \otimes v_2$ determines the first column.

Question: Put the basis vectors $u_i \otimes v_j$ in the correct order so that the above is the matrix of $f \otimes g$

What is the trace of $f \otimes g$?

Definition 1.13 If A, B are rings, we make $A \otimes_{\mathbb{Z}} B$ into a ring by
$$(a \otimes b) \cdot (a_1 \otimes b_1) := aa_1 \otimes bb_1 .$$

Example 1.14 Are any of
$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}, \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}, \quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$$
isomorphic as rings?

dim 4 over $\mathbb{R}$.

A Yes

B No.

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$$
$$r \longmapsto 1 \otimes r$$
$$ab \longleftarrow a \otimes b$$

The tensor product on the left is over Z but if A and B are algebras over a commutative ring R, we can take the tensor product over R.

Pre-class Warm-up!!

Which (if any) of the following matrices CANNOT be expressed as a tensor product of two smaller matrices?

A $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 3 & 6 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

C $\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

D $\dashrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

E They all can

Adjoint property of Tensor Product and Hom

The set-up: if R and S are rings, and (S,R)-bimodule is a left S-module A that is also a right R-module, in such a way that the actions of R and S commute: (ra)s = r(as) always.

$r \in S \quad s \in R.$

If A is an (S,R)-bimodule, B is a left S-module and C is a left R-module then ${}_S A_R \otimes_R {}_R C$ is a left S-module with action given by

$s(a \otimes_R c) = sa \otimes c$

Hom_S(A,B) is a left R-module with action given by $(r\phi)(a) := \phi(ar)$

and Hom_S(B,A) is a right R-module with action given by $(\phi r)(b) := \phi(b)r$

The operation of tensor product on bimodules is associative.

What piece of theory allows us to say there is a left action of S on the tensor product? Is it obvious?

Example: Given a ring hom $f: R \to S$, S is an (S,R)-bimodule

Jump straight to

Theorem 1.16. Let A be an (S,R)-bimodule, B a left S-module and C a left R-module. Then

$$\boxed{\mathrm{Hom}_S(A \otimes_R C, B) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_S(A,B))}$$

via an isomorphism that is natural in B and C. N

Proof. We define mappings $\rightarrow$, $\leftarrow$ that are inverse.

$$\phi \longmapsto (c \mapsto (a \mapsto \phi(a \otimes c)))$$

$$(a \otimes c \mapsto \psi(c)(a)) \longleftarrow\!\!\!| \ \psi$$

We check: $(a,c) \mapsto \psi(c)(a)$ is balanced

The image morphisms are R-linear, S-linear respectively. Check: $\rightarrow$ $\leftarrow$ are inverse. $\square$

$\cdot a = sa \quad a \cdot r = af(r). \quad a, s \in S, r \in R$

Corollary 1.7. Let f : R -> S be a ring homomorphism, let B be a left R-module and let C be a left S-module. Then

$$\mathrm{Hom}_S({}_S S_R \otimes_R B, C) \cong \mathrm{Hom}_R(B, C)$$

Proof.

How is C regarded as an R-module?

$\mathrm{Hom}_S({}_S S_R, C)$ f is a left R-module

$\mathrm{Hom}_S({}_S S_R, C) \cong C$, $\quad f \leftrightarrow f(1_S)$

$$\mathrm{Hom}_S(S \otimes_R B, C) \cong \mathrm{Hom}_R(B, \underbrace{\mathrm{Hom}_S(S, C)}_{\cong C})$$

Question: Is the isomorphism

A only an isomorphism of abelian groups ✓

B an isomorphism of R-modules?

C an isomorphism of S-modules?

Is this

Easy .1 .2 .3 .4 .5 .6 .7 .8 .9 10 Difficult

If you were developing this mathematics for yourself, what is P(you would have thought of it)

Lemma 2.4 of Eisenbud. Let U be a multiplicatively closed subset of a commutative ring R, and M an R-module. Then

$$\underset{R[U^{-1}]}{R[U^{-1}]} \underset{R}{\otimes_R} M \cong M[U^{-1}] \quad \text{as } R[U^{-1}]\text{-modules.}$$

Proof.

1. Define $\rightarrow$ $\leftarrow$ and show they are inverse.

$$\frac{a}{b} \otimes m \longmapsto \frac{am}{b} \qquad b \in U,\ a \in R$$

$$\frac{1}{c} \otimes m \longleftarrow\!\shortmid\ \frac{m}{c}$$

~~0.~~ Checks.

2. Verify $R[U^{-1}] \otimes_R M$ satisfies the universal property that $M[U^{-1}]$

$$\begin{tikzcd} M \arrow[r, "R\text{-modules}"] \arrow[d] & N \quad \sim \quad R[U^{-1}]\text{-module} \\ R[U^{-1}] \otimes M \arrow[ur, "\exists !"'] & \end{tikzcd}$$

$m \mapsto 1 \otimes m$

At one point it probably seemed mysterious why this should be true, and not clear how to prove it. What about now?

Pre-class Warm-up!!!

If you were giving an exposition of localization (e.g. writing a book), how would you define the localization M[U^-1] of a module M at a multiplicative set U ? Would you define it as

A R[U^-1] $\otimes_R$ M

or

B The way it was defined initially in Eisenbud's book.

Discuss some pros and cons of each approach. Bear in mind that whatever you do you are probably going to have to define tensor products anyway.

Why do we define $\otimes$ anyway?

$$\mathbb{Z}^n \hookrightarrow \mathbb{Q}^n \hookrightarrow \mathbb{R}^n$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n = \mathbb{Q}^n$$

$$\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n.$$

Splitting and exactness

Definitions: split mono, split epi, exact, short exact sequence.

A morphism $A \xrightarrow{f} B$ is split mono $\Leftrightarrow \exists\ \phi : B \to A$ so that $\phi f = 1_A$

f is split epi $\Leftrightarrow \exists\ \theta : B \to A$ with $f\theta = 1_B$

A diagram of modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at $B \Leftrightarrow f(A) = \ker g$.

A diagram $0 \to A \to B \to C \to 0$ is a short exact sequence $\Leftrightarrow$ it is exact at A, B and C.

Lemmas: split mono implies mono, split epi implies epi

Proof. f is split mono $\Leftrightarrow \exists\ \phi,\ \phi f = 1_A$

$\Rightarrow$ if $f(u) = f(v)$ then $\phi f(u) = \phi f(v)$

$u = v$ so f is mono $= 1-1$.

split epi is similar! $\square$

If $U \subseteq M$ is a submodule then

$$0 \to U \xrightarrow{\text{incl}} M \xrightarrow{\text{quotient}} M/U \to 0$$

is a s.e.s

Exact at U: $0 = \ker(\text{incl})$ ✓

Exact at M/U: Image of quotient $= \ker(M/U \to 0)$ ✓

Exact at M: $\ker(M \to M/U) = U$

$\overset{?}{=}$ image(incl) $= U$ ✓

Generally $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a s.e.s $\Leftrightarrow B/f(A) \cong C$.

Which of the following are short exact sequences?

A $\quad 0 \longrightarrow R \xrightarrow[\text{left}]{\begin{bmatrix}1\\0\end{bmatrix}} R\oplus R \xrightarrow[\text{right}]{[0\ 1]} R \longrightarrow 0$

B $\quad 0 \longrightarrow R \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} R\oplus R \xrightarrow{[1\ 0]} R \longrightarrow 0$

C $\quad 0 \longrightarrow R \xrightarrow{[1]} R \longrightarrow 0 \longrightarrow 0$

D $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{[5]} \mathbb{Z} \xrightarrow{(1\mapsto \bar{1})} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

E $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \xrightarrow{(1\mapsto \bar{1})} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

? only possibility is 0

Which of them have the left hand map split mono?

Which of them have the right hand map split epi?

	ses?	split mono	split epi.
A	✓	same ✓	same ✓
B	No	✓	
C	✓	✓	✓ Contention.
D	No	No	
E	✓	No.	No

Why is $\mathbb{Z} \xrightarrow[=f]{[5]} \mathbb{Z}$ not split mono?

If it were, $\exists\, \phi : \mathbb{Z} \to \mathbb{Z}$ with $\phi\,[5] = 1_{\mathbb{Z}}$? No.

$\phi \cdot [5]$ has image $\subseteq 5\mathbb{Z}$.

Lemma 2.2. Equivalent for a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

1. f is split mono

2. g is split epi

3. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[dr, "i_1"'] & B \arrow[r, "g"] \arrow[d, "\alpha"] \arrow[l, bend left, "\phi"] & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur, "pr_2"'] & &
\end{tikzcd}$$

where α is an isomorphism

The notation went wrong here. See the next page

Proof. $1 \Rightarrow 3$

Suppose f is split mono so $\exists$ $\phi : B \to A$ with $\phi f = 1_A$.

Define $C_1 = \ker(\phi)$

$3 \Rightarrow 1$. $(pr_1)\alpha f = 1_A$.

see next page also I missed some terms

Definition. If any of the conditions hold we say the short exact sequence is split.

We show $B = f(A) \oplus C_1$.

We check $f(A) \cap C_1 = 0$

If $u = f(v) \in f(A) \cap C_1$.

Then $\phi(u) = \phi f(v) = 0$
$= v$, so $v = 0$.

We check $f(A) + C_1 = B$,

To see this, $f(A) + C_1$ contains C_1 and corresponds to a submodule of B/C_1 via ϕ. It corresponds to $\phi f(A) + C_1 = A + C_1 = B/C_1 = \mathrm{Im}\,\phi \subseteq A$. Thus $f(A) + C_1 = B$, because B corresponds to B/C_1.

Also $C_1 \cong C$ via g b/c $A \cong \ker g$. Let $\alpha : f(A) \oplus C_1 \to C_1 \xrightarrow{\cong} C$

The diagram commutes. $\square$

The last part of (1) ⇒ (3) had notational errors. Here is a better version.

To show $f(A)+C_1 = B$:

$f(A)+C_1$ is a submodule of B that contains C_1, so corresponds to a submodule of $\phi(B)$, by the correspondence theorem.

Now $\phi(f(A)+C_1) = \phi f(A) = A$

$\subseteq \phi(B) \subseteq A$

so that $\phi(f(A)+C_1) = \phi(B)$.

Therefore $f(A)+C_1 = B$, because they both correspond to the same submodule of A.

We deduce: $B = f(A) \oplus C_1$.

We see that the restriction of $g: B \to C$ to C_1 is an isomorphism because $f(A) = \ker(g)$ and g is onto.

Define $\alpha : B = f(A) \oplus C_1 \xrightarrow{\begin{bmatrix} \phi & 0 \\ 0 & g \end{bmatrix}} A \oplus C$.

α is an isomorphism, and the diagram commutes.

Pre-class Warm-up

Let K[x] be the polynomial ring in a variable x with coefficients in a field K. Let J be the ideal J = (x^4).

1. Is 0 -> J -> K[x] -> K[x]/J -> 0 a short exact sequence?

A Yes ✓

B No

2. Is it split as a sequence of K[x]-modules?

A Yes

B No ✓

Discuss!! What is your reasoning?

Homework 2 is now online (Canvas site, my home page) due Thursday in 2 weeks.

$J = $ span of $x^4, x^5, \ldots$

$K[x]/J$ has basis $\bar{1}$ $\bar{x}$ $\bar{x}^2$ $\bar{x}^3$

Note that $\bar{x}^3$ is annihilated by x. If the sequence were split $K[x] \cong J \oplus K[x]/J$ as $K[x]$-modules

No non-zero element of $K[x]$ is annihilated by x, so there is no such iso.

Lemma 2.4 A, B, C, M are left R-modules, N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to Hom_R(C,M) \xrightarrow{g^*} Hom_R(B,M) \xrightarrow{f^*} Hom_R(A,M)$

is exact for all M, if and only if

$N\otimes_R A \to N\otimes_R B \to N\otimes_R C \to 0$

is exact for all N.

2. $0 \to A \to B \to C$ is exact if and only if

$0 \to Hom(M,A) \to Hom(M,B) \to Hom(M,C)$

is exact for all M.

~~Proof~~ 1. Suppose $A \to B \to C \to 0$ is exact (at B and C). We show the sequence of Homs is exact.

Exactness at $Hom(C,M)$ means g^* is 1-1. Suppose $\phi_1, \phi_2 : C \to M$ with $g^*\phi_1 = \phi_1 g = g^*\phi_2 = \phi_2 g$

Question: do we even understand how the morphisms between Homs are constructed? Were they ever defined?

It should say: Let $A \to B \to C \to 0$ be a diagram of R-modules.

f^* is 'precomposition with f'

Given $\theta : B \to M$, $f^*(\theta) = \theta f : A \to M$

---

g is onto, so every element of C can be written $g(b)$ $b \in B$.

$\phi_1 g(b) = \phi_2 g(b)$ $\forall b \in B$.

so $\phi_1(c) = \phi_2(c)$ $\forall c \in C$

Thus $\phi_1 = \phi_2$, g^* is 1-1.

Next: exactness at $Hom(B,M)$.

We show $Im\, g^* \subseteq ker\, f^*$

and $Im\, g^* \supseteq ker\, f^*$ from top

IDEA: We know $gf = 0$, so $f^* g^* \overset{!}{=} (gf)^* = 0$ so $Im\, g^* \subseteq ker\, f^*$

Lemma 2.4 A, B, C, M are left R-modules, N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to \mathrm{Hom}(C,M) \xrightarrow{g^*} \mathrm{Hom}(B,M) \xrightarrow{f^*} \mathrm{Hom}(A,M)$

is exact for all M, if and only if

$N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$

is exact for all N.

2. $0 \to A \to B \to C$ is exact if and only if

$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C)$

is exact for all M.

We show $\mathrm{Im}\, g^* \supseteq \ker f^*$.

Suppo $\alpha : B \to M \in \ker f^*$

so $\alpha f : A \to M$ is 0.

α is 0 on $\mathrm{Im} f$ so defines

a homom $\bar{\alpha} : B/\mathrm{Im} f) \to M$.

Now $\mathrm{Im} f = \ker g$ so

$B/\mathrm{Im} f \cong C$.

We get $\bar{\alpha} : C \to M$.

and $\alpha = \bar{\alpha} g = g^*(\bar{\alpha}) \in \mathrm{Im}\, g^*$.

Done.

---

1. Show $A \to B \to C \to 0$ exact $\Rightarrow$ $\otimes$ sequence is exact. It is exact

$\Leftrightarrow$ $0 \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes C, M) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes B, M) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes A, M)$

is exact $\forall M$.

$\Leftrightarrow$ $0 \to \mathrm{Hom}_R(N, \mathrm{Hom}(C,M)) \to \mathrm{Hom}_R(N, \mathrm{Hom}(B,M)) \to \mathrm{Hom}(N, \mathrm{Hom}(A,M))$ is exact

$\overset{\text{part 2}}{\Leftrightarrow}$ $0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$ is exact $\forall M$

$\overset{\text{part 1}}{\Leftrightarrow}$ $A \to B \to C \to 0$ is exact.

Lemma 2.4 A, B, C, M are left R-modules,
N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$

is exact for all M, if and only if

$N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ is exact $\forall N$.

i.e. sequence of Homs exact $\Rightarrow$ $A \to B \to C \to 0$ exact.

Proof "$\Leftarrow$" We show

g is onto: Let $M = C/g(B)$ quotient

The quotient homomorphism $C \to C/g(B)$ maps to 0 in $\mathrm{Hom}(B,M)$ why?
so must be 0. Thus $C = g(B)$.

$\ker g \subseteq f(A)$:
Take $M = B/f(A)$. The quotient $B \to B/f(A)$ maps to 0 in $\mathrm{Hom}(A,M)$ why?
Thus it is the image of a map $\theta: C \to B/f(A)$.

The quotient factors
$$\begin{tikzcd} B \arrow[r, "q"] \arrow[d, "g"'] & B/f(A) \\ C \arrow[ur, "\theta"'] & \end{tikzcd}$$

If $u \in \ker g$ then $q(u) = \theta g(u) = 0$ so $u \in \ker q = f(A)$.

$f(A) \subseteq \ker g$:
Let $M = C$. The map $g: B \to C$ is in the image of $\mathrm{Hom}(C,C) \to \mathrm{Hom}(B,C)$
namely, the image of 1 why?
Thus g maps to 0 in $\mathrm{Hom}(A,C)$,
$gf = 0$.

Definition 2.5. The functors Hom(- ,M) and Hom(M, -) are left exact, while $N \otimes_R -$ is right exact.

A covariant functor F is exact if and only if

Example.

Definition 2.6. The R-module P is projective if and only if given any diagram

Lemma 2.7 TFAE for an R-module P :

1. P is projective.
2. Every epimorphism M -> P splits.
3. There is a module Q such that $P \oplus Q$ is free.
4. Hom(P, -) is an exact functor.

Definition 2.8. Injective and flat modules.

Proposition 2.9. Projective modules are flat.

Proposition 2.5 of Eisenbud. R[U^-1] is flat as an R-module.

Proof.