Tensor products

In section 2.2 of his book, Eisenbud only does tensor products over a commutative ring, adopting a definition that does not apply in generality, so we have to follow a different source.

In my notes I denote a noncommutative ring by R, and it would be more suitable if the ring were called A, for instance.

We learn (for starters):

The definition of the tensor product of a right R-module M with a left R-module N. $M \otimes_R N$
What a tensor is.
What the tensor product looks like in special cases.
The tensor product of two matrices.
The adjoint property of Hom and Tensor Product.

$(0+0)\otimes n = 0\otimes n = 0\otimes n + 0\otimes n$

$0 = 0\otimes n$

Definition 2.1.1 (my notes) Given a right R-module M, a left R-module N, we construct the free abelian group T with basis (the elements of) $M\times N$

We construct the subgroup S generated by elements

$(m_1+m_2, n) - (m_1, n) - (m_2, n)$

$(m, n_1+n_2) - (m, n_1) - (m, n_2)$

$(mr, n) - (m, rn) \quad \forall\, m \in M, n \in N, r \in R$

Defn. $M\otimes_R N = T/S$

Basic tensors. Defn $m\otimes_R n := (m,n)+S$. An element of $M\otimes_R N$ is a tensor. An arbitrary tensor is a $\mathbb{Z}$-linear combination of basic tensors

Distributivity etc of tensors. The following hold:

$(m_1+m_2)\otimes_R n = m_1\otimes_R n + m_2\otimes_R n$

$m\otimes(n_1+n_2) = m\otimes_R n_1 + m\otimes_R n_2$

$mr\otimes n = m\otimes rn$ always.

Question: how difficult is it to see that $m\otimes 0 = 0 = 0\otimes n$ always?

Scale Easy 1 2 . . . 10 difficult

Definition. M is a right R-module, N a left R-module, L an abelian group.
A mapping f : M x N -> L is R-balanced if and only if

$$f(m_1+m_2, n) = f(m_1, n) + f(m_2, n)$$

$$f(m, n_1+n_2) = f(m, n_1) + f(m, n_2)$$

$$f(mr, n) = f(m, rn)$$

<u>Example</u>. We have a mapping

$$f : M \times N \to M \otimes_R N$$

$$f(m,n) = m \otimes_R n$$

If R is commutative an R-bilinear map $g : M \times N \to L$ an R-module has

$$g(r_1 m_1 + r_2 m_2, n) = r_1 g(m_1, n) + r_2 g(m_2, n)$$

$$g(m, r_1 n_1 + r_2 n_2) = \text{similar.}$$

Such g is balanced.

Discussion. What is the difference between the notion of being balanced and some concept of R-bilinear?

Is it a problem that f(rm,~~s~~n) = rf(m,sn) = rsf(m,n) = srf(m,n)?

Yes

No

Theorem (Dummit and Foote Cor.11)
The balanced map $M \times N \to M \otimes_R N$ is universal with respect to balanced maps.
The tensor product $M \otimes_R N$ is defined up to isomorphism by this property.

Given a balanced map β

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[dr, "f"'] & L \\ & M \otimes_R N \arrow[u, "\alpha"'] \end{tikzcd}$$

$\exists$ unique homom of ab. gps
$M \otimes_R N \xrightarrow{\alpha} L$
so that the Δ commutes.

Theorem. (D & F Theorem 10).
Balanced maps $M \times N \to L$ biject with group homomorphisms $M \otimes_R N \to L$.

<u>Proof</u>. Given a group homom $M \otimes_R N \to L$ we get a balanced map

$$M \times N \xrightarrow{f} M \otimes_R N \to L$$

Given a balanced map

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr] & L \\ & M \otimes_R N \arrow[u, "\alpha"'] \end{tikzcd}$$

$\exists$ unique homom

The two constructions are inverse. $\square$

Pre-class Warm-up!!

True or false?:

The mapping $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathbb{Q}$

specified by $a \otimes_{\mathbb{Z}} b \longmapsto (a, b)$

is a group homomorphism.

A True

B False ✓

$$a \otimes_{\mathbb{Z}} b + c \otimes_{\mathbb{Z}} d \longmapsto (a,b) + (c,d)$$
$$= (a+c, b+d) \overset{?}{=}$$

$$0 \otimes 0 \longmapsto (0,0)$$
$$\parallel \qquad\qquad \neq$$
$$10 \otimes 0 \longmapsto (10, 0)$$

How can construct well-defined homomorphisms?

Is the assignment

$$\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$$
$$(a,b) \to (a,b) \qquad \mathbb{Z}\text{-balanced?}$$

$$? \ (ar, b) \to (ar, b)$$
$$\neq$$
$$(a, rb) \to (a, rb)$$

go to the same place.

Not $\mathbb{Z}$-balanced, don't get a homom.

Examples.

1. If f : R -> S is a ring homomorphism with f(1_)R) = 1_S then $S \otimes_R R \cong S$ as left S-modules.

Example: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$

We regard S as a right R-module via f

True or false?: $S \otimes_R R \cong S$

Proof that

The mapping

Define a mapping $g: S \times R \to S$

$(s,r) \longmapsto s f(r)$

specified by

Claim: this is balanced!

is a group homomorphism.

$g(s_1+s_2, r) = g(s_1,r) + g(s_2,r)$

A True

$g(s, r_1+r_2) = g(s,r_1) + g(s,r_2)$

B False

$g(st, r) = g(s,tr)$ when $t \in R$.

i.e. $g(sf(t), r) = g(s,tr) = s f(tr)$

$= s f(t) f(r)$

How is $S \otimes_R R$ a left S-module?

Defn. $s_1(s \otimes r) := s_1 s \otimes r$ This is well defined because $S \times R \to S \otimes_R R$

$(s,r) \to s_1 s \otimes_R r$ is balanced.

e.g. $(st, r) \to s_1 s f(t) \otimes_R r = s_1 s \otimes_R tr$

Let M be an R-module.

2. Let I be a right ideal of R. Then

$(I \backslash R) \otimes_R M \cong M / IM$

Proof Construct isomorphisms

$H I \otimes m \leftarrow m + IM$ (well-defined)

$(r+I) \otimes m \longrightarrow rm + IM$

using $I \backslash R \times M \longrightarrow M/IM$ is balanced

$(r+I, m) \longrightarrow rm + IM$

These are inverse, get an isomorphism

3. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

We have $S \otimes_R R \to S$. We also construct $S \otimes_R R \xleftarrow{h} S$: h a map of S-modules

$s \otimes_R 1_R \longleftarrow s$

Check $hg = 1_{S \otimes_R R}$ $gh = 1_R$

Conclude $S \otimes_R R \cong S$.

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Proof. $m\mathbb{Z}$ is an ideal in $\mathbb{Z}$

so it is $(\mathbb{Z}/n\mathbb{Z}) / m\mathbb{Z}(\mathbb{Z}/n\mathbb{Z})$

$$\cong \mathbb{Z}/(n\mathbb{Z}+m\mathbb{Z})$$

$$= \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Pre-class Warm-up!

True or false?

The mapping $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$

specified by $a \otimes_{\mathbb{Z}} b \longmapsto ab$

is a group homomorphism.

A True ✓

B False.

There is also a map
$1 \otimes_{\mathbb{Z}} q \longleftarrow q$

$$1 \otimes q + 1 \otimes q = 1 \otimes (q+q)$$
$$= 1 \otimes 2q$$

The composite $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \leftarrow\rightarrow \mathbb{Q}$ is the identity.

What about the composite $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \leftarrow\rightarrow \mathbb{Q}$ is

$$\frac{r}{s} \otimes \frac{u}{v} \longrightarrow 1 \otimes_{\mathbb{Z}} \frac{ru}{sv}.$$

$$= r \otimes \frac{u}{sv}$$

$$s\left(1 \otimes \frac{ru}{sv}\right) = 1 \otimes \frac{ru}{v} = r \otimes \frac{u}{v}$$

Deduce that $1 \otimes \frac{ru}{sv} = \frac{1}{s}\left(r \otimes \frac{u}{v}\right)$

$$= \frac{r}{s} \otimes \frac{u}{v}.$$

Thus $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$

Theorem 1.8.
Tensor product distributes over direct sums:

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$

and similarly on the other side.

Proof. Define $\longrightarrow$ and $\longleftarrow$ that are inverse.

$(M \oplus M') \times N \to$ stuff on right.

$((m, m'), n) \to (m \otimes n, m' \otimes n)$

It's balanced

$(m, 0) \otimes n + (0, m') \otimes n' \longleftarrow (m \otimes_R n, m' \otimes n')$

check balanced for these

Question: Should this be easy or difficult to prove?

Easy 1 . . . 10 Difficult.

Question: Does this seem a reasonable proof?

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$.

Tensor products of vector spaces

Example 1.10 Let U and V be vector spaces over a field K with bases $u_1, \ldots, u_m$, $v_1, \ldots, v_n$.

Then $U \otimes_K V$ is a vector space with basis $u_i \otimes v_j$ $\quad 1 \le i \le m$, $1 \le j \le n$.

so $\dim(U \otimes_K V) = \dim U \cdot \dim V$.

This is because

$$U \otimes V = \left(\bigoplus K u_i\right) \otimes \left(\bigoplus K v_j\right)$$

$$= \bigoplus_{i,j} \left(K u_i \otimes_K K v_j\right)$$

$$= \bigoplus_{i,j} K(u_i \otimes v_j)$$

Definition 1.11. Let f : M -> M' and g : N -> N' be homomorphisms of right and left R-modules. We define

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$$

to be the group homomorphism determined by the balanced map $M \times N \longrightarrow M' \otimes_R N'$ given by $(m,n) \longmapsto f(m) \otimes g(n)$

$(f \otimes g)(m \otimes n) := f(m) \otimes g(n)$

Example 1.12

Check $(mr, n) \longmapsto f(mr) \otimes g n$

$= f(m) \cdot r \otimes g(n)$

$= f(m) \otimes r g(n)$

$= f(m) \otimes g(rn)$

$(m, rn) \longmapsto f(m) \otimes g(rn)$

Let $R = K$ be a field $M = K^2 = N$

$f : K^2 \to K^2$ has matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

$g : K^2 \to K^2$ $\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix}$

Then we get $f \otimes g : K^2 \otimes K^2 \to K^2 \otimes K^2$

has matrix :

$$\left[\begin{array}{cc} 1\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} & 0\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} \\ 2\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} & 3\begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 0 & 5 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 10 & 0 & 15 \\ 8 & 12 & 12 & 18 \end{bmatrix}$$

This is the tensor product of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 4 & 6 \end{bmatrix}$.

Suppose the basis of M is are u_1, u_2 and of N is v_1, v_2.

A $u_1 \otimes v_1,\ u_2 \otimes v_2,\ u_1 \otimes v_2,\ u_2 \otimes v_1$

B ✓ $u_1 \otimes v_1,\ u_1 \otimes v_2,\ u_2 \otimes v_1,\ u_2 \otimes v_2$

C $u_1 \otimes v_1,\ u_2 \otimes v_1,\ u_1 \otimes v_2,\ u_2 \otimes v_2$

D None of the above!

$u_1 \otimes v_1 \longmapsto (u_1 + 2u_2) \otimes 4v_2$

$= 4u_1 \otimes v_2 + 8u_2 \otimes v_2$ determines the first column.

Question: Put the basis vectors $u_i \otimes v_j$ in the correct order so that the above is the matrix of $f \otimes g$

What is the trace of $f \otimes g$?

Definition 1.13 If A, B are rings, we make $A \otimes_{\mathbb{Z}} B$ into a ring by

$$(a \otimes b) \cdot (a_1 \otimes b_1) := aa_1 \otimes bb_1 .$$

Example 1.14 Are any of $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$ isomorphic as rings?

dim 4 over $\mathbb{R}$.

A Yes

B No.

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$$

$$r \longmapsto 1 \otimes r$$

$$ab \longleftarrow a \otimes b$$

The tensor product on the left is over Z but if A and B are algebras over a commutative ring R, we can take the tensor product over R.

Pre-class Warm-up!!

Which (if any) of the following matrices CANNOT be expressed as a tensor product of two smaller matrices?

A $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 3 & 6 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

C $\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

D $\dashrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

E They all can

Adjoint property of Tensor Product and Hom

The set-up: if R and S are rings, and (S,R)-bimodule is a left S-module A that is also a right R-module, in such a way that the actions of R and S commute: (ra)s = r(as) always.

$r \in S$ $s \in R$.

If A is an (S,R)-bimodule, B is a left S-module and C is a left R-module then ${}_S A_R \otimes_R {}_R C$ is a left S-module with action given by

$s(a \otimes_R c) = sa \otimes c$

Hom_S(A,B) is a left R-module with action given by $(r\phi)(a) := \phi(ar)$

and Hom_S(B,A) is a right R-module with action given by $(\phi r)(b) := \phi(b)r$

The operation of tensor product on bimodules is associative.

What piece of theory allows us to say there is a left action of S on the tensor product? Is it obvious?

Example: Given a ring hom $f : R \to S$, S is an (S,R)-bimodule

Jump straight to

Theorem 1.16. Let A be an (S,R)-bimodule, B a left S-module and C a left R-module. Then

$$\boxed{\mathrm{Hom}_S(A \otimes_R C, B) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_S(A,B))}$$

via an isomorphism that is natural in B and C. N

Proof. We define mappings $\rightarrow$, $\leftarrow$ that are inverse.

$\phi \longmapsto (c \mapsto (a \mapsto \phi(a \otimes c)))$

$(a \otimes c \mapsto \psi(c)(a)) \longleftarrow\!\shortmid \psi$

We check: $(a,c) \mapsto \psi(c)(a)$ is balanced. The image morphisms are R-linear, S-linear respectively. Check: $\rightarrow$ $\leftarrow$ are inverse. $\square$

$\cdot a = sa$ $\quad a \cdot r = a f(r)$. $a, s \in S, r \in R$

Corollary 1.7. Let f : R -> S be a ring homomorphism, let B be a left R-module and let C be a left S-module. Then

$$\mathrm{Hom}_S({}_S S_R \otimes_R B, C) \cong \mathrm{Hom}_R(B, C)$$

Proof.

How is C regarded as an R-module?

$\mathrm{Hom}_S({}_S S_R, C)$ is a left R-module

$\mathrm{Hom}_S \cong C$

$f \leftrightarrow f(1_S)$

$$\mathrm{Hom}_S(S \otimes_R B, C) \cong \mathrm{Hom}_R(B, \underbrace{\mathrm{Hom}_S(S, C)}_{\cong C})$$

Question: Is the isomorphism

A only an isomorphism of abelian groups ✓

B an isomorphism of R-modules?

C an isomorphism of S-modules?

Is this

Easy .1 .2 .3 .4 .5 .6 .7 .8 .9 .10 Difficult

If you were developing this mathematics for yourself, what is P(you would have thought of it)

Lemma 2.4 of Eisenbud. Let U be a multiplicatively closed subset of a commutative ring R, and M an R-module. Then

$${}_{R[U^{-1}]}R[U^{-1}]_{R} \otimes_R M \cong M[U^{-1}] \text{ as } R[U^{-1}]\text{-modules.}$$

At one point it probably seemed mysterious why this should be true, and not clear how to prove it. What about now?

Proof.

1. Define $\rightarrow$ $\leftarrow$ and show they are inverse.

$$\frac{a}{b}\otimes m \longmapsto \frac{am}{b} \qquad b \in U,\ a \in R$$

$$\frac{1}{c}\otimes m \longleftarrow\!\!\shortmid \frac{m}{c}$$

Checks.

Or 2. Verify $R[U^{-1}]\otimes_R M$ satisfies the universal property that $M[U^{-1}]$ ...

$$\begin{tikzcd} M \arrow[r, "R\text{-modules}"] \arrow[d] & N \arrow[r, phantom, "\sim"] & R[U^{-1}]\text{-module} \\ R[U^{-1}]\otimes M \arrow[ur, "\exists !"'] & & \end{tikzcd}$$

$m \mapsto 1\otimes m$

Pre-class Warm-up!!!

If you were giving an exposition of localization (e.g. writing a book), how would you define the localization M[U^-1] of a module M at a multiplicative set U ? Would you define it as

A R[U^-1] $\otimes_R$ M

or

B The way it was defined initially in Eisenbud's book.

Discuss some pros and cons of each approach. Bear in mind that whatever you do you are probably going to have to define tensor products anyway.

Why do we define $\otimes$ anyway?

$$\mathbb{Z}^n \hookrightarrow \mathbb{Q}^n \hookrightarrow \mathbb{R}^n$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n = \mathbb{Q}^n$$

$$\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n .$$

Splitting and exactness

Definitions: split mono, split epi, exact, short exact sequence.

A morphism $A \xrightarrow{f} B$ is split mono $\Leftrightarrow \exists\ \phi : B \to A$ so that $\phi f = 1_A$

f is split epi $\Leftrightarrow \exists\ \theta : B \to A$ with $f\theta = 1_B$

A diagram of modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at $B \Leftrightarrow f(A) = \ker g$.

A diagram $0 \to A \to B \to C \to 0$ is a short exact sequence $\Leftrightarrow$ it is exact at A, B and C.

Lemmas: split mono implies mono, split epi implies epi

Proof. f is split mono $\Leftrightarrow \exists\ \phi,\ \phi f = 1_A$

$\Rightarrow$ if $f(u) = f(v)$ then $\phi f(u) = \phi f(v)$

$u = v$ so f is mono $= 1\text{-}1$.

split epi is similar! $\square$

If $U \subseteq M$ is a submodule then

$$0 \to U \xrightarrow{\text{incl}} M \xrightarrow{\text{quotient}} M/U \to 0$$

is a s.e.s

Exact at U: $0 = \ker(\text{incl})$ ✓

Exact at M/U Image of quotient $= \ker(M/U \to 0)$ ✓

Exact at M: $\ker(M \to M/U) = U$

$\overset{?}{=}$ image(incl) $= U$ ✓

Generally $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a s.e.s $\Leftrightarrow B/f(A) \cong C$.

Which of the following are short exact sequences?

A $\quad 0 \to R \xrightarrow[\text{left}]{\begin{bmatrix}1\\0\end{bmatrix}} R\oplus R \xrightarrow[\text{right}]{[0\ 1]} R \to 0$

B $\quad 0 \to R \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} R\oplus R \xrightarrow{[1\ 0]} R \to 0$

C $\quad 0 \to R \xrightarrow{[1]} R \to 0 \to 0$

D $\quad 0 \to \mathbb{Z} \xrightarrow{[5]} \mathbb{Z} \xrightarrow{(1\mapsto \bar{1})} \mathbb{Z}/2\mathbb{Z} \to 0$

E $\quad 0 \to \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \xrightarrow{(1\mapsto \bar{1})} \mathbb{Z}/2\mathbb{Z} \to 0$

? only possibility is 0

Which of them have the left hand map split mono?

Which of them have the right hand map split epi?

	ses?	split mono	split epi.
A	✓	same ✓	same ✓
B	No	✓	
C	✓	✓	✓ Contention.
D	No	No	
E	✓	No.	No

Why is $\mathbb{Z} \xrightarrow[=f]{[5]} \mathbb{Z}$ not split mono?

If it were, $\exists\, \phi : \mathbb{Z} \to \mathbb{Z}$ with $\phi\,[5] = 1_{\mathbb{Z}}$? No.

$\phi \cdot [5]$ has image $\subseteq 5\mathbb{Z}$.

Lemma 2.2. Equivalent for a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

1. f is split mono

2. g is split epi

3. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, shift left, "f"] \arrow[dr, "i_1"'] & B \arrow[l, shift left, "\phi"] \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur, "pr_2"'] & &
\end{tikzcd}$$

where α is an isomorphism

The notation went wrong here. See the next page

Proof. $1 \Rightarrow 3$

Suppose f is split mono so $\exists$ $\phi : B \to A$ with $\phi f = 1_A$.

Define $C_1 = \ker(\phi)$

$3 \Rightarrow 1$. $(pr_1)\alpha f = 1_A$.

See next page also I missed some terms

Definition. If any of the conditions hold we say the short exact sequence is split.

We show $B = f(A) \oplus C_1$.

We check $f(A) \cap C_1 = 0$

If $u = f(v) \in f(A) \cap C_1$.

Then $\phi(u) = \phi f(v) = 0$
$= v$, so $v = 0$.

We check $f(A) + C_1 = B$,

To see this, $f(A) + C_1$ contains C_1 and corresponds to a submodule of B/C_1 via ϕ. It corresponds to $\phi f(A) + C_1 = A + C_1 = B/C_1$
$= \operatorname{Im}\phi \subseteq A$. Thus $f(A) + C_1 = B$, because B corresponds to B/C_1.

Also $C_1 \cong C$ via g b/c $A \cong \ker g$.

Let $\alpha : f(A) \oplus C_1 \to C_1 \xrightarrow{\cong} C$

The diagram commutes. $\square$

The last part of (1) ⇒ (3) had notational errors. Here is a better version.

To show $f(A)+C_1=B$:

$f(A)+C_1$ is a submodule of B that contains C_1, so corresponds to a submodule of $\phi(B)$, by the correspondence theorem.

Now $\phi(f(A)+C_1) = \phi f(A) = A$

$\subseteq \phi(B) \subseteq A$

so that $\phi(f(A)+C_1) = \phi(B)$.

Therefore $f(A)+C_1 = B$, because they both correspond to the same submodule of A.

We deduce: $B = f(A)\oplus C_1$.

We see that the restriction of $g: B \to C$ to C_1 is an isomorphism because $f(A) = \ker(g)$ and g is onto.

Define $\alpha: B = f(A)\oplus C_1 \xrightarrow{\begin{bmatrix}\phi & 0\\ 0 & g\end{bmatrix}} A\oplus C$.

α is an isomorphism, and the diagram commutes.

Pre-class Warm-up

Let K[x] be the polynomial ring in a variable x with coefficients in a field K. Let J be the ideal J = (x^4).

1. Is 0 -> J -> K[x] -> K[x]/J -> 0 a short exact sequence?

A Yes ✓

B No

2. Is it split as a sequence of K[x]-modules?

A Yes

B No ✓

Discuss!! What is your reasoning?

Homework 2 is now online (Canvas site, my home page) due Thursday in 2 weeks.

J = span of $x^4, x^5, \ldots$

$K[x]/J$ has basis $\bar{1}\quad \bar{x}\quad \bar{x}^2\quad \bar{x}^3$

Note that $\bar{x}^3$ is annihilated by x. If the sequence were split $K[x] \cong J \oplus K[x]/J$ as $K[x]$-modules

No non-zero element of $K[x]$ is annihilated by x, so there is no such iso.

Lemma 2.4 A, B, C, M are left R-modules, N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to Hom_R(C,M) \xrightarrow{g^*} Hom_R(B,M) \xrightarrow{f^*} Hom_R(A,M)$

is exact for all M, if and only if

$N\otimes_R A \to N\otimes_R B \to N\otimes_R C \to 0$

is exact for all N.

2. 0 -> A -> B -> C is exact if and only if

0 -> Hom(M,A) -> Hom(M,B) -> Hom(M,C)

is exact for all M.

Proof 1. Suppose $A \to B \to C \to 0$ is exact (at B and C). We show the sequence of Homs is exact.

Exactness at Hom(C, M) means g^* is 1-1. Suppose $\phi_1, \phi_2 : C \to M$ with $g^*\phi_1 = \phi_1 g = g^*\phi_2 = \phi_2 g$

Question: do we even understand how the morphisms between Homs are constructed? Were they ever defined?

It should say: Let $A \to B \to C \to 0$ be a diagram of R-modules.

f^* is 'precomposition with f'

Given $\theta : B \to M$, $f^*(\theta) = \theta f : A \to M$

---

g is onto, so every element of C can be written $g(b)$ $b \in B$.

$\phi_1 g(b) = \phi_2 g(b)$ $\forall b \in B$.

so $\phi_1(c) = \phi_2(c)$ $\forall c \in C$

Thus $\phi_1 = \phi_2$, g^* is 1-1.

Next: exactness at Hom(B, M).

We show $Im\, g^* \subseteq \ker f^*$ and $Im\, g^* \supseteq \ker f^*$

IDEA: from top

We know $gf = 0$, so $f^* g^* \overset{!}{=} (gf)^* = 0$ so $Im\, g^* \subseteq \ker f^*$

Lemma 2.4 A, B, C, M are left R-modules, N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to \mathrm{Hom}(C,M) \xrightarrow{g^*} \mathrm{Hom}(B,M) \xrightarrow{f^*} \mathrm{Hom}(A,M)$

is exact for all M, if and only if

$N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$

is exact for all N.

2. $0 \to A \to B \to C$ is exact if and only if

$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C)$

is exact for all M.

We show $\mathrm{Im}\, g^* \supseteq \ker f^*$.
Suppo $\alpha : B \to M \in \ker f^*$
so $\alpha f : A \to M$ is 0.
α is 0 on $\mathrm{Im} f$ so defines
a homom $\bar{\alpha} : B/\mathrm{Im} f) \to M$.
Now $\mathrm{Im} f = \ker g$ so
$B/\mathrm{Im} f \cong C$.
We get $\bar{\alpha} : C \to M$.
and $\alpha = \bar{\alpha} g = g^*(\bar{\alpha}) \in \mathrm{Im}\, g^*$.
Done.

---

1. Show $A \to B \to C \to 0$ exact $\Rightarrow$ $\otimes$ sequence is exact. It is exact

$\Leftrightarrow$ $0 \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes C, M) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes B, M) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes A, M)$

is exact $\forall M$.

$\Leftrightarrow$ $0 \to \mathrm{Hom}_R(N, \mathrm{Hom}(C,M)) \to \mathrm{Hom}_R(N, \mathrm{Hom}(B,M)) \to \mathrm{Hom}(N, \mathrm{Hom}(A,M))$ is exact

$\overset{\text{part 2}}{\Leftrightarrow}$ $0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$ is exact $\forall M$

$\overset{\text{part 1}}{\Leftrightarrow}$ $A \to B \to C \to 0$ is exact.

Lemma 2.4 A, B, C, M are left R-modules,
N is a right R-module.

1. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if

$0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$

is exact for all M, if and only if
$N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ is exact $\forall N$.

Proof "$\Leftarrow$" i.e. sequence of Homs exact $\Rightarrow$ We show $A \to B \to C \to 0$ exact.

g is onto: Let $M = C/g(B)$ quotient

The quotient homomorphism $C \to C/g(B)$ maps to 0 in $\mathrm{Hom}(B,M)$ why? so must be 0. Thus $C = g(B)$.

$\ker g \subseteq f(A)$:
Take $M = B/f(A)$. The quotient $B \to B/f(A)$ maps to 0 in $\mathrm{Hom}(A,M)$ why? Thus it is the image of a map $\theta: C \to B/f(A)$.

The quotient factors
$$\begin{tikzcd} B \arrow[r, "q"] \arrow[dr, "g"'] & B/f(A) \\ & C \arrow[u, "\theta"'] \end{tikzcd}$$

If $u \in \ker g$ then $q(u) = \theta g(u) = 0$ so $u \in \ker q = f(A)$.

$f(A) \subseteq \ker g$:
Let $M = C$. The map $g: B \to C$ is in the image of $\mathrm{Hom}(C,C) \to \mathrm{Hom}(B,C)$ namely, the image of 1 why?
Thus g maps to 0 in $\mathrm{Hom}(A,C)$,
$gf = 0$.

Pre-class Warm-up

Consider the short exact sequence

$0 \to Z \xrightarrow{2} Z \to Z/2Z \to 0.$

We can apply $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$ to each of its terms to get a sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{iso}} \mathbb{Z}/2\mathbb{Z} \to 0$$

1. Is this sequence exact?

is exact here

Yes

No ✓

2 The same question with Hom. Is the following sequence exact?

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$$

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Yes ✓

exact here

No

Class on Friday 10/15 and Monday 10/18 will be online via Zoom. See my home page or the Canvas site for the Zoom link.

Pre-class Warm-up!!

Consider the following two functors

F, G : abelian groups -> abelian groups

$F(M) = \mathbb{Z}^2 \otimes_{\mathbb{Z}} M$

$\mathbb{Z} \otimes_{\mathbb{Z}} M \cong M$

$\mathbb{Z}^2 \otimes_{\mathbb{Z}} M \cong M \oplus M$

$G(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, M)$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \cong M$

$(1 \mapsto m) \leftrightarrow m$

$G(M) = M \oplus M$. Same deal: G is exact.

1. How easy is it to see whether or not F is exact?

Easy 1 2 3 4 5 6 7 8 9 10 Difficult

2. How easy is it to see whether or not G is exact?

What about $\mathbb{Q} \otimes_{\mathbb{Z}} M$? This is exact

$\mathbb{Q} \otimes_{\mathbb{Z}} -$ is always right exact.

If $0 \to A \xrightarrow{f} B$ ~~had~~ f mono is

$\mathbb{Q} \otimes_{\mathbb{Z}} A \xrightarrow{1 \otimes f} \mathbb{Q} \otimes B$ mono?

Yes.

s.e.s $\mathbb{Z}$

$F(0 \to A \to B \to C \to 0)$

$= 0 \to A \oplus A \to B \oplus B \to C \oplus C \to 0$ is also exact

$\mathbb{Z}^2 \otimes M \cong M \oplus M$

$(a,b) \otimes m \longleftrightarrow (am, bm)$ $\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$

$(a,b) \otimes f(m) \longleftrightarrow (af(m), bf(m))$

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Definition 2.5. The functors Hom(- ,M) and Hom(M, -) are left exact, while $N \otimes_R -$ is right exact.

contravariant

covariant

A covariant functor F is left exact, right exact, exact if and only if

A functor F: R-modules $\longrightarrow$ abelian groups

a specification: for each R-module M an abelian group $F(M)$

$\forall$ R-module homoms $M \xrightarrow{f} M'$

a homom. $F(f): F(M) \to F(M')$

Example. $F = \mathrm{Hom}_R(U, -)$

here if $f: M \to M'$

$F(f) = f_* : \mathrm{Hom}_R(U, M) \to \mathrm{Hom}(U, M')$

F must satisfy $F(fg) = F(f) \circ F(g)$

$F(1_M) = 1_{F(M)}$.

F is left exact $\Leftrightarrow$ $\forall$ exact diagrams $0 \to A \to B \to C$

$0 \to F(A) \to F(B) \to F(C)$ is exact.

Definition 2.6. The R-module P is projective if and only if given any diagram with an exact row (i.e. g is epi)

$$\begin{tikzcd} & P \arrow[d, "\alpha"] \arrow[dl, dashed, "\exists\theta"'] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

then $\exists\, \theta: P \to B$ so that $\alpha = g\theta$

Lemma 2.7 TFAE for an R-module P :

1. P is projective.
2. Every epimorphism $M \to P$ splits.
3. There is a module Q such that $P \oplus Q$ is free.
4. Hom(P, -) is an exact functor.

Proof 1 $\Rightarrow$ 2. Assume 1. let $M \xrightarrow{g} P$ be epi. Consider

$$\begin{tikzcd} & P \arrow[d, "1"] \arrow[dl, "\theta"'] \\ M \arrow[r, "g"] & P \end{tikzcd}$$

$\exists\, \theta$ so that $1 = g\theta$. Thus g is split epi.

$2 \Rightarrow 3$. Suppose every epi splits. $g: M \to P$

We show

P is a direct summand of R^n.

Take a set of generators for P

giving a surjection $R^n \xrightarrow{g} P$

$n = \#$ generators.

This splits. $R^n \cong P \oplus \ker(g)$.

Not done directly.

$\boxed{(3) \Rightarrow (4)}$ Suppose $R^n = P \oplus Q$

We show $\mathrm{Hom}_R(P, -)$ is exact.

To do this, we only need to show

If $B \xrightarrow{g} C$ is epi then $\mathrm{Hom}(P,B) \xrightarrow{g^*} \mathrm{Hom}(P,C)$ is epi.

This happens $\iff$ given $\alpha \in \mathrm{Hom}(P,C)$

$\exists\ \theta \in \mathrm{Hom}(P,B)$ with $g_*(\theta) = \alpha$

$g\theta = \alpha$

$\iff$ condition 1.

We see $(4) \iff (1)$.

In fact we do $(3) \Rightarrow (1)$.

Suppose $R^n = P \oplus Q$ and consider

$$
\begin{tikzcd}
 & R^n \arrow[dl, "\hat{\theta}"'] \arrow[d, shift left, "\pi"] & \\
 & P \arrow[u, shift left, "i"] \arrow[d, "\alpha"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}
$$

We find $\hat{\theta}: R^n \to B$ so that

$g\hat{\theta} = \alpha\pi$

For each basis element e_j of R^n

find $b_j \in B$ so that $g(b_j) = \alpha\pi(e_j)$

The assignment $e_j \mapsto b_j\ \forall j$

extends to $\hat{\theta}$. (R^n is projective.)

Define $\theta = \hat{\theta} i : P \to B$.

Then $g\theta = g\hat{\theta} i = \alpha\pi i = \alpha 1_P = \alpha$

F is right exact $\iff$ $\forall$ exact diagrams $A \to B \to C \to 0$

$F(A) \to F(B) \to F(C) \to 0$ is exact

F is exact $\iff$ F is both right and left exact.

Example $\mathrm{Hom}(U,-)$ is left exact
$N \otimes -$ is right exact.

F is contravariant if $F(fg) = F(g)F(f)$

We say such F is left exact $\iff$ $\forall$ exact diagrams $A \to B \to C \to 0$
we have $0 \to F(C) \to F(B) \to F(A)$
is exact.
left here

Example $\mathrm{Hom}(-,U)$ is left exact.

Definition. An functor abelian groups -> abelian groups is linear if and only if

Proposition. TFAE for a linear covariant functor F.

1. F is both right and left exact.
2. For every s.e.s 0 -> A -> B -> C -> 0, the sequence 0 -> F(A) -> F(B) -> F(C) -> 0 is exact.
3. For every exact sequence ...-> A -> B -> C ->... the sequence ... -> F(A) -> F(B) -> F(C) -> ... is exact

Attempt to prove $1 \Rightarrow 3$.

Write the exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & \mathrm{Im} f \arrow[ur] \arrow[dr] & \\ 0 \arrow[ur] & & 0 \end{tikzcd}$$

and are exact at Im f.

Apply F

$$\begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr] & & F(B) \arrow[dr] & \\ & F(\mathrm{Im} f) \arrow[ur, "G"] \arrow[dr] & & F(\mathrm{Im} g) \arrow[dr] \\ 0 \arrow[ur] & & 0 & & 0 \end{tikzcd}$$

by 1. also exact

We show $\mathrm{Im}\, F(f) = \ker F(g)$

Application: For a chain complex

(C.,d.)

we have H_i(F(C.)) = F(H_i(C.))

$0 \to F(\mathrm{Im} f) \to F(B) \to F(\mathrm{Im}\, g) \to 0$

is a s.e.s by (2)

Also $F(\mathrm{Im} f) = \mathrm{Im}\, F(f)$ by commutativity of

$$\begin{tikzcd} \cdot \arrow[rr] \arrow[dr] & & \cdot \\ & \cdot \arrow[ur, "G"'] & \end{tikzcd}$$

$F(\mathrm{Im} f) = \ker\left(F(B) \to F(\mathrm{Im}\, g)\right)$

$= \ker F(g)$.

$= \ker F(B) \to F(\mathrm{Im}\, g)$.

Definition 2.8. Injective and flat modules.

I is injective $\Leftrightarrow$ $\mathrm{Hom}_R(-, I)$ is an exact functor

A right R-module N is flat $\Leftrightarrow$ $N \otimes_R -$ is exact.

Proposition 2.9. Projective modules are flat.

Proof.

Step 1 R is flat

$$R \otimes_R M \cong M$$

Step 2 R^n is flat

$$R^n \otimes M \cong \underbrace{M \oplus \cdots \oplus M}_{n}$$

Step 3. If $P \oplus Q \cong R^n$ then P is flat. If $A \to B$ is mono then so is

$$(P \oplus Q) \otimes (A \to B)$$

$$\cong \begin{bmatrix} P \otimes A \\ \oplus \\ Q \otimes A \end{bmatrix} \longrightarrow \begin{bmatrix} P \otimes B \\ \oplus \\ Q \otimes B \end{bmatrix}$$

and on each summand $P \otimes A \to P \otimes B$ we have mono.

Pre-class Warm-up

If you were trying to determine whether $\mathbb{Z}$ is injective as an abelian group, which of the following 3 criteria for injectivity would you choose to use?

A The diagrammatic definition. Given

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u, "\alpha"] \arrow[r, "f"] & B \arrow[ul, dashed, "\psi"'] \end{tikzcd}$$

with f mono, $\exists \psi : B \to I$ so that $\alpha = \psi f$

B Every monomorphism $I \to B$ is split mono.

C Hom(- , I) is exact.

D Something else.

$\mathbb{Z}$ is not injective

Option B: Find a mono $\mathbb{Z} \to B$ that is not split mono

e.g. $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is not split mono $\mathbb{Z} \not\cong 2\mathbb{Z} \oplus$ something.

Option C: Is $\mathrm{Hom}(-, \mathbb{Z})$ exact?

Consider $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

Apply $\mathrm{Hom}(-, \mathbb{Z})$ to get

$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to 0$

which is not exact.

Option A:

$$\begin{tikzcd} & \mathbb{Z} & \\ 0 \arrow[r] & \mathbb{Z} \arrow[u, "1"] \arrow[r, "2"] & \mathbb{Z} \arrow[ul, dashed, "\exists ?"'] \end{tikzcd}$$

No.

Proposition 2.5 of Eisenbud. R[U^-1] is flat as an R-module. Localization is an exact Functor.

Proof.

We verify that $R[U^{-1}] \otimes_R -$ is exact. Recall $M[U^{-1}] \cong R[U^{-1}] \otimes_R M$.

We know: $\otimes$ is right exact.

We check: if we have a mono

$0 \to M \to M_1$ then

$0 \to M[U^{-1}] \to M_1[U^{-1}]$ is exact.

Assume M is a submodule of M_1.

Suppose $\frac{m}{u} \in M[U^{-1}]$ goes to 0 in $M_1[U^{-1}]$. This means $\exists v \in U$ with $vm = 0$ in M_1.

This relation also holds in M, so $\frac{m}{u} = 0$ in M. $\square$

Proposition. Let
F : abelian groups -> abelian groups be a linear exact functor.
Let $M_1, M_2, \dots, M_t$
be submodules of an R-module M. Then
$F(M_1 \cap \dots \cap M_t) = F(M_1) \cap \dots \cap F(M_t)$.

Here F exact $\Rightarrow$ if $M_1 \subseteq M$ then $F(M_1)$ can be regarded as a submodule of $F(M)$.

Proof. There is an exact sequence

$$0 \to M_1 \cap \dots \cap M_t \to M \xrightarrow{\begin{bmatrix} q_1 \\ \vdots \\ q_t \end{bmatrix}} M/M_1 \oplus \dots \oplus M/M_t$$

$q_i : M \to M/M_i$ is the quotient map.

F is exact so

$$0 \to F(M_1 \cap \dots \cap M_t) \to F(M) \xrightarrow{\alpha} F(M/M_1) \oplus \dots \oplus F(M/M_t)$$

$\ker \alpha = \bigcap F(M_i) = F(M_1 \cap \dots \cap M_t)$

b/c $\ker F(M) \to F(M/M_i) \cong F(M_i)$

from $0 \to F(M_i) \to F(M) \to F(M/M_i) \to 0$ is a s.e.s.

Set up:

R is a commutative ring

$U \subseteq R$ is mult. closet.

Localization is

$R[U^{-1}] \otimes_R - : R\text{-modules} \to R[U^{-1}]\text{-modules}$

It is exact means

$\forall$ s.e.s. of R-modules $0 \to A \to B \to C \to 0$

the sequence

$$0 \to R[U^{-1}] \otimes_R A \to R[U^{-1}] \otimes_R B \to R[U^{-1}] \otimes_R C \to 0$$

is exact.

Definition. The support of an R-module M is the set of prime ideals P of R for which $M_P \neq 0$.

Corollary 2.7 of Eisenbud. If M is finitely generated, P is a prime ideal of R, then P lies in the support of M if and only if P contains the annihilator of M.

Other defn.

$$\operatorname{Supp} M = \{\text{prime ideals } P \mid P \supseteq \operatorname{ann}_R(M)\}$$
$$\longleftrightarrow \operatorname{Spec}(R/\operatorname{ann}_R(M)).$$

Proof. $m \in M$ with $\frac{m}{1} \neq 0$ in $M_P \iff m$ is not annihilated by $R - P = U \iff \operatorname{ann}(m) \subseteq P$.

$\exists$ such $m \iff \operatorname{ann}(M) \subseteq \operatorname{ann}(m) \subseteq P$.

$\square$

M is non-zero $\iff \operatorname{ann}_R M \neq R$.

$\iff \operatorname{ann}_R M \subseteq \mathfrak{m}$ some maximal ideal.

$\iff \exists\, \mathfrak{m} \in \operatorname{supp}(M)$

True or false?:

1. If M is a non-zero R-module then the support of M is non-empty.

A True ✓

B False

2. How difficult is it to establish your claim in 1?

A Easy

B Moderately easy ✓

C Moderately hard

D Quite hard

A similar question to the last one.

Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ are two maps of R-modules.
True or false?
If $gf \neq 0$ then there exists a maximal ideal $\mathfrak{m}$ so that the composite $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is non-zero.

A True ✓

B False

This question

What about: if $gf \neq 0$ then for all maximal ideals $\mathfrak{m}$ the composite is non-zero

A true

B false.

It could happen that $\mathrm{Im}(gf)_{\mathfrak{m}} = 0$ in which case $(gf)_{\mathfrak{m}} = 0$.

Pre-class Warm-up!

Consider $\mathrm{Im}(gf)$ so $A \xrightarrow{gf} \cancel{C} \to \mathrm{Im}(gf) \to 0$ is exact

find a max ideal $\mathfrak{m}$ so that $\mathrm{Im}(gf)_{\mathfrak{m}} \neq 0$

Now $A_{\mathfrak{m}} \xrightarrow{(gf)_{\mathfrak{m}}} \cancel{C_{\mathfrak{m}}} \to \underset{\neq 0}{\mathrm{Im}(gf)_{\mathfrak{m}}} \to 0$ is exact

so $(gf)_{\mathfrak{m}} \neq 0$.

A stronger property than flatness

Corollary 2.9 A diagram of R-modules

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

is a short exact sequence if and only if for every maximal ideal $\mathfrak{m}$ the localized sequence

$$0 \to A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}} \to 0$$

is exact.

We know "$\Rightarrow$"

We do "$\Leftarrow$" on the next page.

This uses:

Lemma 2.8 Let m be an element of an R-module M. If m goes to 0 in each localization of M at a maximal ideal of R then $m = 0$.

$M = 0$ if and only if all its localizations at maximal ideals are 0.

Proofs. Suppose $0 \to A_m \to B_m \to C_m \to 0$ exact. $\forall m$

To show $B \xrightarrow{g} C$ is onto, suppose not, let $B \xrightarrow{g} C \to D \to 0$ be exact, so $D = \operatorname{coker} g \neq 0$. Find $m \in \operatorname{supp} D$ so $B_m \to C_m \to D_m \to 0$ is exact, $D_m \neq 0$, $B_m \to C_m$ is not onto contradiction

To show $f(A) \subseteq \ker(g)$: if not, then $gf \neq 0$ so $A_m \to B_m \to C_m$ is not zero for some m. contradiction.

To show $f(A) \supseteq \ker g$ consider the s.e.s.

$0 \to A \to \ker(g) \to E \to 0$. Then

$0 \to A_m \to \ker(g)_m \to E_m \to 0$ is exact

and $\exists\, m$ with $E_m \neq 0$. if $E \neq 0$

contradiction. because $\operatorname{Im}(f_m) = \ker(g_m)$

To show $A \xrightarrow{f} B$ is 1-1 consider the s.e.s

$0 \to \ker f \to A \to f(A) \to 0$.

Then $0 \to (\ker f)_m \to A_m \to f(A)_m \to 0$ is exact and $A_m \to f(A)_m$ is iso $\forall m$

so $(\ker f)_m = 0$ $\forall m$ and $\ker f = 0$.

Thus f is 1-1. $\square$

Another property of localization

Proposition 2.10 Let S be an R-algebra. Let M, N be R-modules with M finitely presented. If S is flat over R then the homomorphism

$$S\otimes_R \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(S\otimes_R M, S\otimes_R N)$$

is an isomorphism.

Proof Do it 1. When M = R
2. When M = R^n
3. When there is an exact sequence

$R^m \to R^n \to M \to 0$. This is what M finitely presented means, with m, n finite.

1. Let $M = R$. $\mathrm{Hom}_R(R,N) \cong N$

$S\otimes_R R \cong S$. The statement is

$$S\otimes_R N \longrightarrow \mathrm{Hom}_S(S, S\otimes_R N) \text{ is iso.}$$

$$\mathrm{Hom}_S(S, S\otimes_R N) = S\otimes_R N$$

✓

The homomorphism is

$$(f: M\to N) \longrightarrow \left(\begin{array}{c} S\otimes M \to S\otimes N \\ 1\otimes m \mapsto 1\otimes f(m)\end{array}\right)$$

$$\lambda\otimes f \longmapsto (1\otimes m \mapsto \lambda\otimes f(m))$$

---

2. The map commutes with $\oplus$

e.g. $\mathrm{Hom}(R^n,N) = \mathrm{Hom}(\bigoplus^n R, N)$
$\cong \bigoplus^n \mathrm{Hom}(R,N) \cong \bigoplus^n N = N^n$.

3. Apply both functors to the exact sequence using $\mathrm{Hom}(\ ,N)$ is left exact, $S\otimes_R -$ is exact. We get a diagram

$$\begin{tikzcd}
 & ? & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & S\otimes \mathrm{Hom}(M,N) \arrow[r] \arrow[d] & S\otimes\mathrm{Hom}(R^n,N) \arrow[r] \arrow[d, "\cong"] & S\otimes \mathrm{Hom}(R^m,N) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(S\otimes M, S\otimes N) \arrow[r] & \mathrm{Hom}(R^n, S\otimes N) \arrow[r] \arrow[d] & \mathrm{Hom}(R^m, S\otimes N) \arrow[d] \\
 & ? & 0 & 0
\end{tikzcd}$$

with exact rows. Two RH vertical maps are iso. Deduce the LH vertical map is iso.
We deduce ?, ? are 0 using :

The Snake Lemma.

Suppose we have a diagram of modules

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker\alpha \arrow[d] & \ker\beta \arrow[d] & \ker\gamma \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}\alpha \arrow[d] & \operatorname{coker}\beta \arrow[d] & \operatorname{coker}\gamma \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

in which the rows are exact. Then there is a long exact sequence.

see purple.

Furthermore, if f is mono or g' is epi then the first map in the l.e.s is mono, or the last map is epi.

$$\ker\alpha \to \ker\beta \to \ker\gamma$$

$$\operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

Given the diagram we introduce extra terms to make columns exact.

The long ex. seq is

$$\ker\alpha \xrightarrow{f} \ker\beta \xrightarrow{g} \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \xrightarrow{f'} \operatorname{coker}\beta \xrightarrow{g'} \operatorname{coker}\gamma$$

we get

$$0 \to \ker\alpha \to \ker\beta \ \cdots$$

or

$$\operatorname{coker} \xrightarrow{g'} \operatorname{coker}\gamma \to 0.$$

How to use the Snake Lemma to prove part 3

Take a presentation of M: $R^m \to R^n \to M \to 0$

Construct.

$$\begin{tikzcd}
0 \arrow[r] & S \otimes \mathrm{Hom}(M,N) \arrow[r] \arrow[d, "\alpha_M"] & S \otimes \mathrm{Hom}(R^n,N) \arrow[r] \arrow[d, "\alpha_{R^n}"] & S \otimes \mathrm{Hom}(R^m,N) \arrow[d, "\alpha_{R^m}"] \\
0 \arrow[r] & \mathrm{Hom}(S \otimes M, S \otimes N) \arrow[r] & \mathrm{Hom}(S \otimes R^n, S \otimes N) \arrow[r] & \mathrm{Hom}(S \otimes R^m, S \otimes N)
\end{tikzcd}$$

Replace the two right hand terms by cokernels to get rows that are s.e.s.

Now use the long e.s. to deduce α_M is iso.