2.4 The structure of finite length modules and commutative rings.

Proposition. Every simple R-module M is isomorphic to R/I for some maximal (left) ideal I. Every such quotient R/I where I is a maximal (left) ideal is a simple R-module. When R is commutative, R/I and R/J are isomorphic simple modules if and only if I = J.

simple = irreducible.

Proof. If M is simple, let $0 \neq x \in M$.
There is an R-module homom.

$\alpha : {}_R R \longrightarrow M$ which is non-zero
$r \longmapsto rx$ hence surjective
b/c M is simple

$M \cong R / \text{kernel of } \alpha$

ideals of R containing
ker α
$\longleftrightarrow$ to submodules of M.
M is simple $\Rightarrow$ the only such ideals are ker α and R.
ker α is a max ideal. .

$I = \mathrm{Ann}_R(R/I)$
$= \mathrm{Ann}(R/J)$ if $R/I \cong R/J$.
$= J$.

Pre-class Warm-up!!

Let S be the $\mathbb{C}[x]$-module that is $\mathbb{C}$ as a vector space, with x acting on it as multiplication by 2, and let T be the $\mathbb{C}[x]$-module that is $\mathbb{C}$ as a vector space, with x acting on it as multiplication by 1.

Are S and T isomorphic as $\mathbb{C}[x]$-modules?

A Yes

B No ✓

$\mathrm{Ann}_{\mathbb{C}[x]} S = (x-2)$

$\mathrm{Ann}_{\mathbb{C}[x]} T = (x-1)$

An isomorphism $f: S \to T$ is a linear map so that

$f(x\lambda) = x f(\lambda)$ always

$f(2\lambda) = 1 \cdot f(\lambda)$? No, b/c

$f(2\lambda) = 2 f(\lambda)$ in fact.

Theorem 2.13. An R-module M has a composition series if and only if it is both Noetherian and Artinian. ✓

Let M have a composition series. Then:

1. Every chain of submodules can be refined to a composition series. Math 8202 ✓

2. If R is commutative the map

$$M \longrightarrow \bigoplus_{\substack{\text{max ideals} \\ I \text{ of } R}} M_I$$

is an isomorphism. The maximal ideals I that appear in the direct sum are those for which M has a composition factor R/I, and the length of M_I is the number of such composition factors.

3. M = M_I if and only if M is annihilated by some power of I.

M is Noetherian $\Leftrightarrow$ M has ACC on submodules.

M is Artinian $\Leftrightarrow$ M has DCC on submodules.

A maximal chain of submodules
$0 = M_0 \subsetneq M_1 \subseteq \cdots \subseteq M_t = M$
is a composition series of M.
Maximal $\Leftrightarrow$ all M_i/M_{i-1} are simple.
These are the 'composition factors'
t = Length of M.
'Finite length' = 'has a composition series'

Proof of 2. We show: after localizing the morphism at each max I we get an isomorphism. Localizing at maximal J gives

$$M_J \longrightarrow \boxed{\bigoplus_I (M_I)_J = M_J}$$

with the identity map, which is iso.
First part done!

Added later: more justification needed.

We show: $M_I \neq 0 \iff M$ has a compn factor $\cong R/I$.

Take a s.e.s.

$$0 \to M_1 \to M \to M/M_1 \to 0$$

giving a s.e.s.

$$0 \to (M_1)_I \to M_I \to (M/M_1)_I \to 0$$

so $M_I \neq 0 \iff$ either $(M_1)_I \neq 0$ or $(M/M_1)_I \neq 0$.

By induction Length$((M/M_1)_I)$ $=$ # compn. factors of M/M_1, $\cong R/I$.

Also $(M_1)_I \neq 0 \iff M_1 \cong R/I$

so Length(M_I) $=$ # compn factors of M, $\cong R/I$.

Note: all c.f.s of M_I are $\cong R/I$.

Refine $M \supset IM \supset \dots$ to a comp. series

All simple factors are ann. by I, so are $\cong R/I$

3. $M_J \neq 0 \iff J \supseteq \operatorname{ann} M$.

$M = M_I \iff M_J = 0$ $\forall$ max $J \neq I$

$\iff$ The only max ideal containing $\operatorname{ann} M$ is I

$\iff$ all comp. factors of M are R/I

In this case

if $0 = M_0 \subset M_1 \subset \dots \subset M_t = M$.

then $I \cdot M \subseteq M_{t-1}$

$I^2 M \subseteq I M_{t-1} \subseteq M_{t-2}$

$\vdots$

$I^t M \subseteq M_0 = 0$.

Conversely

$M \supset IM \supset I^2 M \supseteq \dots$

has all factors direct sums of R/I, so the only c.f.s of M are R/I. I is the only max ideal $\supseteq \operatorname{ann} M$.

Can we recall why it is that if I and J are distinct prime ideals then $(M_I)_J = 0$?

with $I \not\subseteq J$
$J \not\subseteq I$.

$$0 \to I \cdot M_I \to M_I \to M_I / I \cdot M_I \to 0$$

is exact and

$$0 \to (I \cdot M_I)_J \to (M_I)_J \to \underset{\substack{\| \\ 0}}{(M_I / I \cdot M_I)_J} \to 0$$

$u \notin I$, $v \notin J$.

$$\frac{m}{u} \in M_I$$

$$\frac{m}{u} \Big/ v = 0 \iff$$

$\exists\ w \notin J$ with $\frac{wm}{u} = 0$

$$\frac{a}{v} = \frac{0}{1} \iff \exists\ w \notin J$$

with $w(a \cdot 1 - v \cdot 0) = 0$

$\iff wa = 0$.

Added later: it's not true in this generality

e.g. $(\mathbb{Z}_{(2)})_{(3)} = \mathbb{Q}$.

Pre-class Warm-up!!

Which of the following are true for a module M over a commutative ring R?

A If M is simple then there is a unique prime ideal in the support of M. True ✓ False

B If M has a unique prime ideal I in its support and J is a prime ideal with J ≠ I then the localization M_J = 0.

C If M is a simple and J is a prime ideal not in the support of M, then the localization M_J = 0. True ✓ False.

D If M has finite length and J is a prime ideal not in the support of M, then the localization M_J = 0. True ✓ False

This is what was needed in 2.13.

$I \not\subseteq J$, $J \not\subseteq I \Rightarrow (M_I)_J = 0$ is not true $(\mathbb{Z}_{(2)})_{(3)} = \mathbb{Q}$.

M simple $\Rightarrow M \cong R/I$ for some max ideal $I = \operatorname{ann} M$. so $\operatorname{supp} M = \{I\}$ = primes containing ann M.

harder? (giving that it is a quick question). Skip to C

Write $M = R/I$, $J \neq I$, $\exists x \in I - J$ x acts invertibly on M_J, but as 0 on M. so $M_J = 0$.

Compn factors of M are R/I where $I \subseteq \operatorname{supp} M$. All comp. factors of M localize to 0 at J, so $M_J = 0$ (by exactness of localization)

Theorem. TFAE for a commutative ring R.
1. R is Noetherian and all prime ideals are maximal.
2. R has finite length as an R-module.
3. R is Artinian.

For non-commutative rings Hopkins proved Artinian implies Noetherian.

We already know (it's easy):

Proposition. For any module M, TFAE
1. M is Noetherian and Artinian.
2. M has finite length as an R-module.

We will show:

Theorem. Let R be a Noetherian and Artinian commutative ring. Then R has finitely many maximal ideals, all prime ideals are maximal, and

$$R \cong \prod_{I \text{ maximal}} R_I.$$

so R is a product of local rings.

Proof. Take M to be R in 2.13

As an R-module $R \to \prod_{\substack{I \text{ max} \\ \text{ideal}}} R_I$

is an iso, finitely many I are involved. Here we have a ring isomorphism.

The max ideals of $A_1 \times \cdots \times A_t$ are $A_1 \times \cdots \times I_j \times \cdots \times A_t$ where I_j is max. in A_j. (Exercise)

so R has fin. many max ideals.

Claim: If $I_1, \ldots, I_d$ are the max ideals then $I_1^{n_1} \cdots I_d^{n_d} = 0$ for some $n_1, \ldots, n_d$.

(If $0 = R_c \subseteq R_{c-1} \subseteq \cdots \subseteq R_0 = R$ is a comp. series, $R_0/R_1 = R/I_1$ then $I_1 R \subseteq R_1$. If $R_1/R_2 = R/I_2$ then $I_2 I_1 R \subseteq R_2$ etc. Eventually the product is 0)

If P is prime then $P \supseteq I_1^{n_1} \cdots I_d^{n_d}$

so $P \supseteq I_j$ for some j, $P = I_j$. $\square$

We also show:

Theorem: If R is Noetherian and all prime ideals are maximal then R has finite length as an R-module.

Proof. The book does this.

Step 1 Assume R does not have finite length and let I be maximal such that R/I is not of finite length. We show I is prime.

Step 2 If such I is maximal then R/I is a field, which is of finite length contradiction.

To show Step 1. Let $ab \in I$, $a \notin I$ $b \notin I$. Consider the s.e.s.

$$0 \to R/\{r \in R \mid ra \in I\} \xrightarrow{a} R/I \to R/I+(a) \to 0$$

The left and right terms have finite length because $I+(b) \subseteq \{r \in R \mid ra \in I\}$ and $I \neq I+(a)$.

Thus R/I has finite length, contradiction.

Or: Use Exercise 1.2 on p47: If R is Noetherian: There are only finitely many primes minimal over I. Thus R has finitely many max ideals. Their intersection is rad(0), which is finitely generated hence nilpotent.

Show $R/\mathfrak{m}_1 \cdots \mathfrak{m}_t$ has finite length, as does rad(0). Hence R has finite length.

Pre-class Warm-up!!

At the end of last class I briefly mentioned the result below.

Corollary 2.17. Let R be a Noetherian ring, and let M be a finitely generated R-module. TFAE

a. M has finite length

b. Some finite product of maximal ideals annihilates M.

c. All the primes that contain the annihilator of M are maximal.

d. R/ann(M) is an Artinian ring.

Question: Which of the following implications have I either done in class, or follow easily from something done in class?

A a implies b

B b implies c

C c implies d

D d implies a No

E c implies a

Another thing: why should we even want to know about Artinian rings, or modules of finite length?

Corollary 2.17. Let R be a Noetherian ring, and let M be a finitely generated R-module. TFAE

a. M has finite length

b. Some finite product of maximal ideals annihilates M.

c. All the primes that contain the annihilator of M are maximal.

d. $R/\mathrm{ann}(M)$ is an Artinian ring.

Corollary 2.18. Let R be a Noetherian ring, $0 \neq M$ a f.g. R-module with annihilator I, P a prime ideal containing I. The R_P-module M_P is a non-zero module of finite length if and only if P is minimal among primes containing I.

Idea: If $P \supseteq I$ (an ideal), P is minimal prime over I then $(R/I)_P$ has P_P as its only prime ideal. (Prime ideals of $(R/I)_P$ $\leftrightarrow$ prime ideals of R/I, contained in P/I)

Corollary 2.19. Let I be an ideal in a Noetherian ring R. TFAE for a prime P containing I.

a. P is minimal among primes containing I.
b. R_P/I_P is Artinian.
c. In the localization R_P we have

$$P_P^n \subseteq I_P \text{ for some } n > 0.$$

One more thing:

Corollary 2.15. Let X be an affine algebraic set over an algebraically closed field k.
TFAE

a. X is finite
b. A(X) is a finite dimensional vector space over k whose dimension is the number of points in X.
c. A(X) is Artinian.

Assume a. If X has n points then $A(X) = \underbrace{k \times \cdots \times k}_{n}$ as a ring.

so b. also c.

c $\Rightarrow$ A(X) has finitely many max. ideals. $\leftrightarrow$ points of X by the Nullstellsatz.

$A(\{p\}) = \{\text{functions } \{p\} \to k\} \leftrightarrow k$

For each of the following statements, how easy is it for you to see if it is True or False?

A If $\cdots \subseteq P_i \subseteq P_{i+1} \subseteq \cdots$ is a chain of prime ideals then $\bigcap_i P_i$ is prime. Yes No

B In a Noetherian ring, rad(0) is nilpotent. Yes No

C If the ring R = R_1 x R_2 is a product of rings and M is any R-module then there is a decomposition

$M = M_1 \oplus M_2$

where R_1 acts as 0 on M_2 and R_2 acts as 0 on M_1.

Yes.
No

Have you seen it before?

Easy 1 10 Difficult.

In R, write $1_R = (1_{R_1}, 1_{R_2})$.
$(1_{R_1}, 0)$, $(0, 1_{R_2})$ are orthogonal idempotents.

Define $M_1 = (1_{R_1}, 0)M$
$M_2 = (0, 1_{R_2})M$

$R_1 = R(1_{R_1}, 0)$ so $R_1 \cdot M_2 = R(1_{R_1}, 0)(0, 1_{R_2})M = 0$

$m = (1_{R_1}, 0)m + (0, 1_{R_2})m \in M_1 + M_2$

If $m \in M_1 \cap M_2$ then $(1_{R_1}, 0)m = 0$ and $(0, 1_{R_2})m = 0$, so $1_R m = 0 = m$.

Pre-class Warm-up!!

Let x be an element of an R-module M. Are any of the following statements necessarily true?

A R has a submodule isomorphic to M/ <x> , where <x> is the submodule generated by x. True False ✗

B R has a quotient module isomorphic to M/ <x>. ✗ Such a quotient can be generated by one element.

C M has a submodule isomorphic to R/ Ann x. ✓

If $x \in M$, define $f : R \to M$, $r \mapsto rx$ an R-module homomorphism.

$f(R) \cong R/\ker f = R/\mathrm{Ann}(x)$.

Here's another:

D: If I is an ideal of R and M has a submodule isomorphic to R/I then there is an element $y \in M$ with $\mathrm{Ann}(y) = I$