Chapter 3 of Eisenbud: Associated primes.

Definition. Let M be an R-module where R is commutative. A prime P of R is associated to M if $\exists\ x \in M$ with $\mathrm{Ann}_R(x) = P$.

$\Leftrightarrow$ M has a submodule $\cong R/P$.

Ass_R(M) or Ass(M) is the set of primes associated to M.

Notice that $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$
$= \{\text{primes } P \text{ so that } M_P \neq 0\} = \{\text{primes } P \supseteq \mathrm{Ann}\, M\}$

Example. $\mathbb{Z}/30\mathbb{Z}$ as a $\mathbb{Z}$-module has $\mathrm{Ass}(\mathbb{Z}/30\mathbb{Z}) = \{(2),(3),(5)\}$

$\mathbb{Z}$ has support $\{\underset{\text{all } p}{(p)}, \{0\}\}$.

$\mathrm{Ass}_{\mathbb{Z}}(\mathbb{Z}) = \{0\}$.

Goal:

Theorem 3.1 R is a Noetherian ring and $M \neq 0$ is a finitely generated R-module.

a. Ass M is finite and non-empty, each containing Ann M. It includes the minimal primes over Ann M.

b. Every zero divisor on M is contained in an associated prime.

c. Forming Ass commutes with localization.

a. $\Rightarrow$ $\{\text{primes minimal over } I\}$ is finite

b. $\Leftrightarrow$ $\{\text{zero divisors on } M\} = \bigcup \mathrm{Ass}(M)$.

Reality check: why do we even want to know about prime ideals in the first place?

Question: how many associated primes does $Z / 12Z$ have?

A 1

B 2

C 3

D Something else.

R is always Noetherian, M is f.g.

Proposition 3.4. If I is an ideal of R maximal among all ideals of R that are annihilators of elements of M, then I is prime, so belongs to Ass M.

Proof. Let $I = \mathrm{Ann}_R(x)$ be max. among annihilators of elts of M.

Let $r, s \in R$ with $rs \in I$.

Suppose $s \notin I$. Consider $sx \neq 0$

Then $r \in \mathrm{Ann}(sx)$ and

$I \subseteq \mathrm{Ann}(sx)$

so $\underbrace{(r) + I}_{\mathrm{Ann}(sx)} \supseteq I$ is a larger annihilator of an element.

Thus $(r) + I = I$, $r \in I$

I is prime. $\square$

Corollary. Ass M is non-empty. 3.1b holds.

3.1b: $\bigcup \mathrm{Ass}\, M = \{\text{zerodivisors in } M\}$

3.4 $\Rightarrow \forall\ x \in M$,

$\mathrm{Ann}(x) \subseteq (\text{some } P \in \mathrm{Ass}(M))$

Showing that Ass M is finite.

Throughout, M denotes a finitely generated module for a Noetherian ring R.

Lemma 3.6.

a. If $M = M' \oplus M''$ then

$\text{Ass}(M) = (\text{Ass } M') \cup (\text{Ass } M'')$

b. If $0 \to M' \to M \to M'' \to 0$ is a s.e.s. then

$\text{Ass } M' \subseteq \text{Ass } M \subseteq (\text{Ass } M') \cup (\text{Ass } M'')$

Proof a. $\Leftarrow$ b. $\Rightarrow \text{Ass}(M') \subseteq M$

so $\text{Ass}(M') \cup \text{Ass}(M'') \subseteq M$

$\supseteq \Leftarrow$ b.

b. Assume $M' \subseteq M$.

If $x \in M'$, $\text{Ann}_R(x) = P$ is prime

the same is true regarding M.

so $\text{Ass}(M') \subseteq \text{Ass}(M)$.

Suppose $P \in \text{Ass}(M) - \text{Ass}(M')$ so

$P = \text{Ann}(y)$ $y \in M$.

M has a submodule $\cong R/P$

$\quad\quad\quad\quad\quad\quad \overset{\parallel}{Ry}$

$\forall\ 0 \neq z \in Ry$, $\text{Ann}(z) = P$.

Thus $z \notin M'$, and $M' \cap Ry = \{0\}$

Thus $Ry \cong$ image of Ry in M''.

Thus $P \in \text{Ass}(M'')$. $\square$

Why is the following true:

If P is a prime ideal then every non-zero R-submodule of R/P has annihilator P.

Is this easy or difficult?

Easy 1 . . . 10 Difficult.

Proposition 3.7. If R is a Noetherian ring and M is a finitely generated R-module, then M has a filtration

$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$

With each $M_{i+1}/M_i \cong P_i$ for some prime ideal P_i