Some homological algebra

See Appendix A3 of Eisenbud, Rotmans' Homological Algebra, and my notes.

Chain Complexes.

We learn: what is a chain complex, it's homology.
Maps of chain complexes, homotopies between maps.
The long exact sequence coming from a short exact sequence of chain complexes.

Definitions. R is a not-necessarily-commutative ring.

A chain complex is a diagram of R-modules

$$\mathcal{M} = \cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

where $d_i d_{i+1} = 0$ always.
Abbreviated $d^2 = 0$. M_i is in degree i

The boundary maps d_i in the chain complex have degree -1. They lower degree by -1.

The homology group in degree i is

$$H_i(\mathcal{M}) := \ker d_i / \operatorname{Im} d_{i+1}$$

Abbreviate $H_*(\mathcal{M}) := \bigoplus_{i \in \mathbb{Z}} H_i(\mathcal{M})$ instead of $H_i(\mathcal{M})\ \forall i$.

A complex $M_{i+1} \xleftarrow{d_i} M_i \xleftarrow{d_{i-1}} M_{i-1}$ is a cochain complex. It has cohomology

$$H^i(\mathcal{M}) = \ker d_i / \operatorname{Im} d_{i-1}.$$

Pre-class Warm-up

Is the following specification a chain complex?

$$\xrightarrow{\alpha} M_0 \xrightarrow{\beta} M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \xrightarrow{\alpha}$$

where $M_i = \mathbb{Z}^2$ for all i,

$$\alpha = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A Yes

B No

This is a cochain complex.

$\alpha : M \to N$

A morphisms of chain complexes, or chain map is a diagram

$$\begin{array}{ccccc} M_{i+1} & \xrightarrow{d_{i+1}} & M_i & \xrightarrow{d_i} & M_{i-1} \\ \downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} \\ N_{i+1} & \xrightarrow{e_{i+1}} & N_i & \xrightarrow{e_i} & N_{i-1} \end{array}$$

of module homomorphism so that every square commutes

$\alpha_{i-1} d_i = e_i \alpha_i \quad \forall i$.

Lemma. A chain map f induces homomorphisms of homology groups. If $f : M \to N$ we get a homom. $H_i(f) : H_i(M) \to H_i(N)$

Construction of $H_i(f)$:

$f_i(\operatorname{Im} d_{i+1}) \subseteq \operatorname{Im}(e_{i+1})$ and

$f_i(\ker d_i) \subseteq \ker e_i$ because

$f_i d_{i+1}(m) = e_{i+1} f_{i+1}(m) \in \operatorname{Im}(e_{i+1})$

If $d_i(m) = 0$ then $f_i(m) \in \ker e_i$ b/c

$e_i f_i(m) = f_{i-1} d_i(m) = 0$.

We define $H_i(f)(m + d_{i+1}(M_{i+1})) = f_i(m) + e_{i+1} N_{i+1}$ where $m \in \ker d_i$ which makes sense and is independent of choice of m.

Example.

$$\begin{array}{ccccccccc} & & \mathbb{Z} & & 0 & & \mathbb{Z} & & H_* \\ 0 & \to & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1&1\\1&1\end{bmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1&-1\\-1&1\end{bmatrix}} & \mathbb{Z}^2 & \to & 0 \\ & & \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{[2]} & \mathbb{Z} & \xrightarrow{[0]} & \mathbb{Z} & \to & 0 \\ & & 0 & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z} & & \\ \text{degrees} & & 2 & & 1 & & 0 & & \end{array}$$

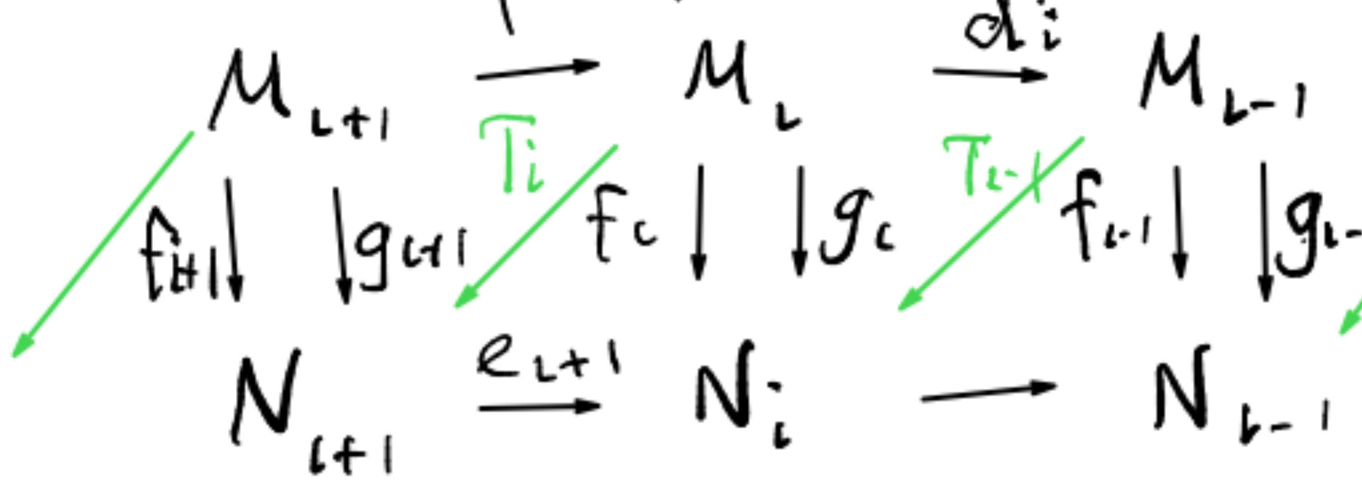
This is a chain map. e.g.

$[2]\,[1 \quad 1] = [1, 1]\begin{bmatrix}1&1\\1&1\end{bmatrix}$

Question: what is the homology of the second complex in degree 2? In degree 1?

A 0 — degree 2
B Z
C Z/2Z — degree 1
D something else

Definition. A chain homotopy between two morphisms is specified as follows:



f is homotopic to g, $f \simeq g$

$\Longleftrightarrow$ $\exists$ $T_i : M_i \to N_{i+1}$ (module homomorphisms) so that

$$f_i - g_i = T_{i-1} d_i + e_{i+1} T_i$$

T is a 'degree $+1$ map'

no commutativity is required.

Proposition. 1. If f and g are homotopic chain maps then the two mappings $H_*(f), H_*(g) : H_*(M) \to H_*(N)$ are the same.

2. If there are chain maps f: $M \to N$ and $g: N \to M$ with $gf \simeq 1_M$ and $fg \simeq 1_N$ then $H_*(f), H_*(g)$ are inverse isomorphisms on homology.

Proof. Suppose $f_i - g_i = T_{i-1} d_i + e_{i+1} T_i$ $\forall i$. Let $m \in \ker d_i \subseteq M_i$

Consider

$f_i(m + \operatorname{Im} d_{i+1}) - g_i(m + \operatorname{Im} d_{i+1})$

$\subseteq (f_i - g_i) m + \operatorname{Im} e_{i+1}$

$= T_{i-1} \underbrace{d_i(m)}_{=0} + e_{i+1} T_i(m) + \operatorname{Im} e_{i+1}$

$= e_{i+1} T_i(m) + \operatorname{Im} e_{i+1} = \operatorname{Im}(e_{i+1})$

$= 0 \in H_i(N)$. Thus $H_i(f) = H_i(g)$

If $f: M \to N$
$g: N \to M$
$gf \simeq 1_M \quad fg \simeq 1_N$
Then $H_*(f): H_*(M) \rightleftarrows H_*(N): H_*(g)$
are inverse isomorphisms.

Proof. $H_*(gf) = H_*(f)\, H_*(g)$

$$= H_*(1_M) = 1_{H_*(M)}$$

Also $H_*(g)\, H_*(f) = 1_{H_*(N)}$.

Thus $H_*(g), H_*(f)$ are inverse isomorphisms. □

Definition Chain complexes M, N are (chain) homotopy equivalent $\Leftrightarrow$ $\exists$ chain maps $f: M \to N$, $g: N \to M$ so that

$$gf \simeq 1_M$$
$$fg \simeq 1_N.$$

Pre-class Warm-up

Consider the two chain complexes with non-zero terms in degrees 0 and 1:

$$
\begin{array}{lccccccc}
 & & & 0 & & \mathbb{Z}/2\mathbb{Z} & & \\
\mathcal{M}: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 \\
 & {\scriptstyle 0}\downarrow & & {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 0} \\
\mathcal{N}: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \longrightarrow & 0 \\
 & & & 0 & & \mathbb{Z}/3\mathbb{Z} & &
\end{array}
$$

1. Are there any non-zero chain maps $\mathcal{M} \to \mathcal{N}$?

Yes

2. Are there any chain maps $\mathcal{M} \longrightarrow \mathcal{N}$ that induce non-zero maps on homology?

No

A Yes

B No

Examples. k is field

Degrees: $2 \quad 1 \quad 0$

1.
$$\begin{array}{ccccccccc}
0 \to (& k & \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} & k^2 & \xrightarrow{\begin{bmatrix}1 & -1\end{bmatrix}} & k &) \to 0 \to 0 \\
f \downarrow & \downarrow 1 & & \downarrow \begin{bmatrix}1&0\\0&1\end{bmatrix} & & \downarrow 1 & \\
0 \to (& k & \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} & k^2 & \xrightarrow{\begin{bmatrix}1 & -1\end{bmatrix}} & k &) \to 0
\end{array}$$

(diagonal maps: $\begin{bmatrix}0 & 1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$)

Consider the degree $+1$ map T.

Calculate $T_0 d_1 + d_2 T_1$

$$= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= f_1$$

$$d_1 T_0 + T_{-1} d_0 = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = [1] = f_0$$

2. $R = k[x]/(x^3)$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
\alpha \downarrow & 0 \downarrow & & \downarrow [x^2] \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
\beta \downarrow & [x^2] \downarrow & & \downarrow 0 \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
0 \downarrow & 0 \downarrow & & \downarrow 0 \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

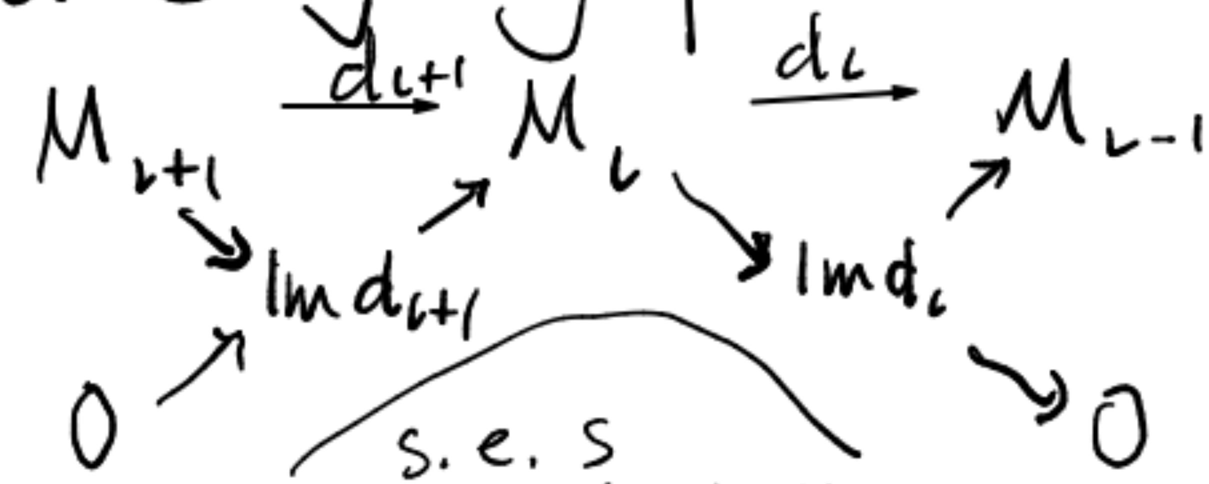
Question. Are any of the chain maps $\alpha, \beta, 0$ chain homotopic? $\alpha \simeq \beta$ $\quad \alpha \not\simeq 0$

$$T_1 d_2 + d_3 T_2 = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1 = f_2.$$

$$f - 0 = Td + dT \text{ so } f \simeq 0.$$

Definition. If the chain complex M has $1_M \simeq 0$ we call M contractible.

Dull Fact. M is contractible $\Leftrightarrow$ it is acyclic (= zero homology = exact everywhere) and "everything splits"



$\Leftrightarrow$ exact at M_i.

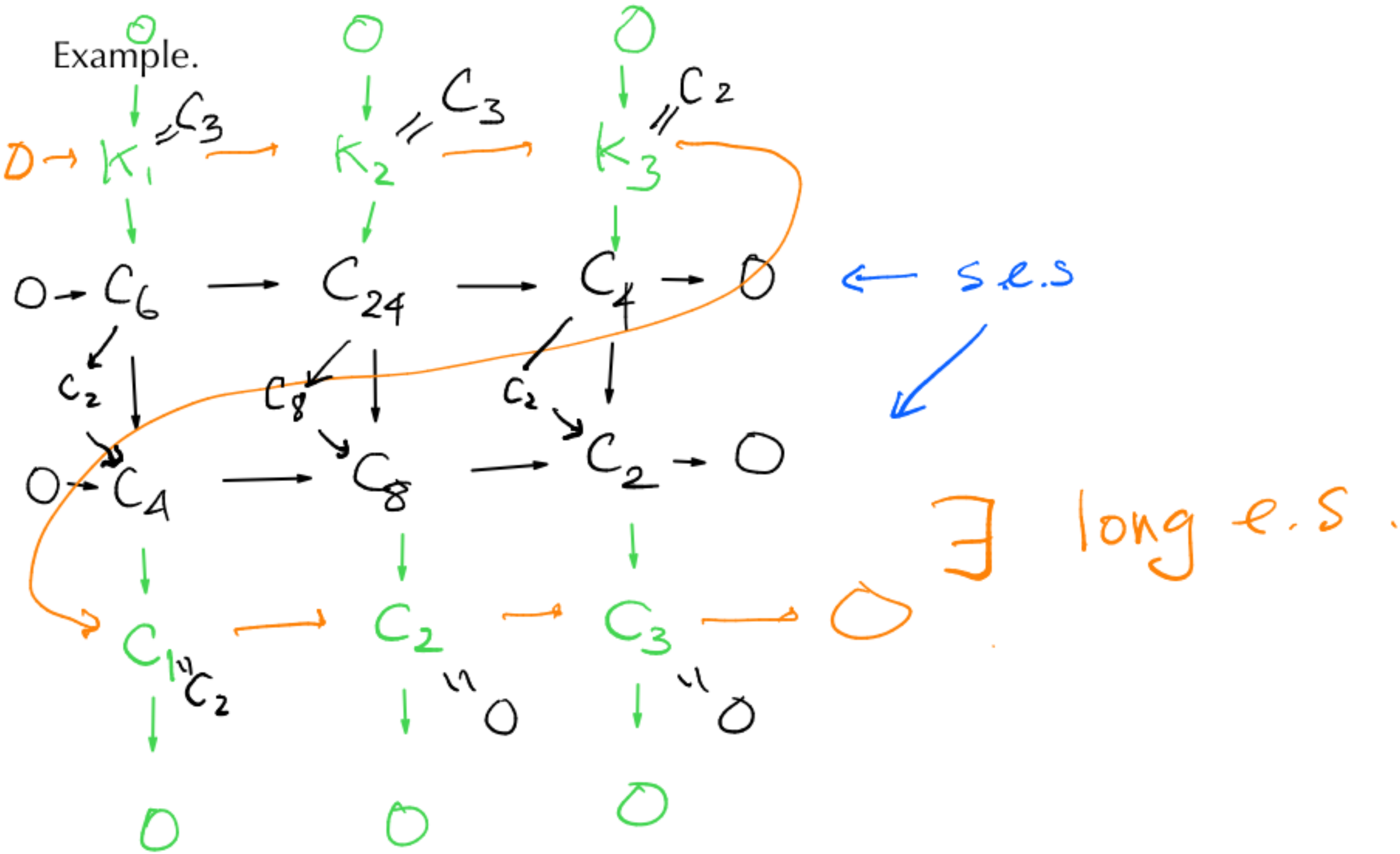
All this s.e.s. split in a contractible cx.

If $X \simeq pt$ is a contractible space then

$$1_{\tilde{C}.(X)} \text{ is contractible.}$$

The snake lemma

Example.

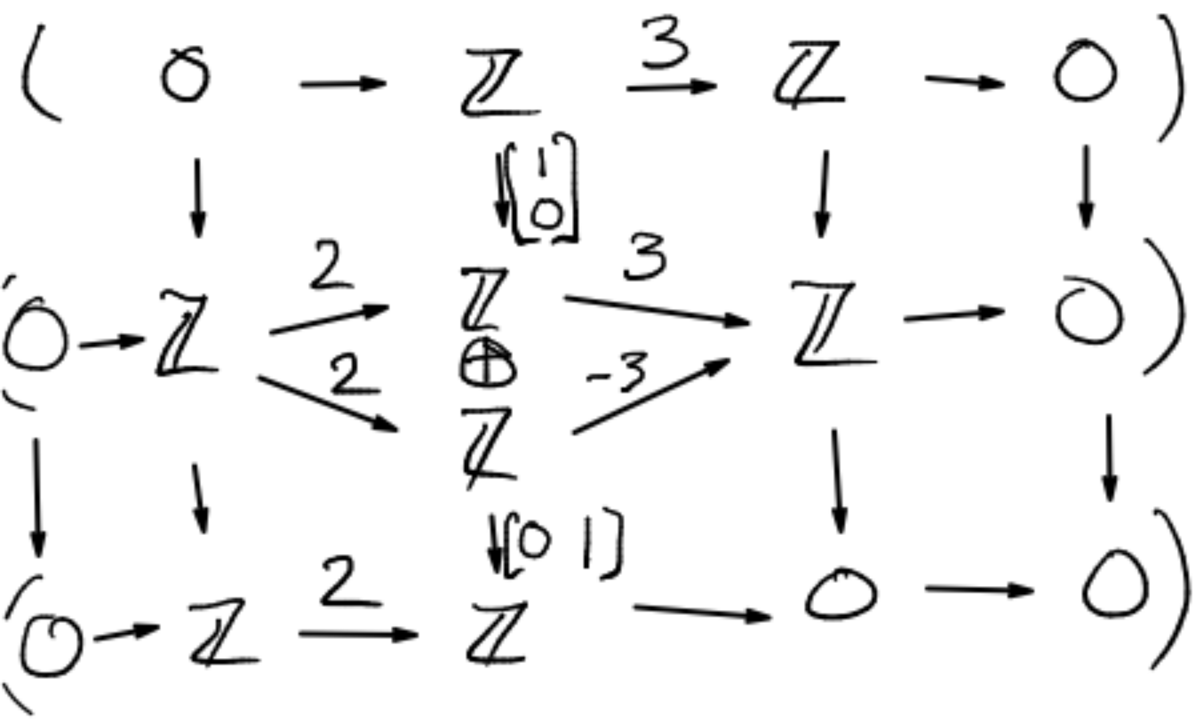


The long exact homology sequence.

Theorem 2.3.6. A short exact sequence of chain complexes $0 \to \mathcal{L} \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{N} \to 0$ gives rise to a long exact sequence in homology.

The connecting homomorphism is natural.

Example:



Projective resolutions