Some homological algebra

See Appendix A3 of Eisenbud, Rotmans' Homological Algebra, and my notes.

Chain Complexes.

We learn: what is a chain complex,
it's homology.
Maps of chain complexes, homotopies between maps.
The long exact sequence coming from a short exact sequence of chain complexes.

Definitions. R is a not-necessarily-commutative ring.

A chain complex is a diagram of R-modules

$$\mathcal{M} = \cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

where $d_i d_{i+1} = 0$ always.
Abbreviated $d^2 = 0$. M_i is in degree i

The boundary maps d_i in the chain complex have degree -1. They lower degree by -1.

The homology group in degree i is

$$H_i(\mathcal{M}) := \ker d_i / \operatorname{Im} d_{i+1}$$

Abbreviate $H_*(\mathcal{M}) := \bigoplus_{i \in \mathbb{Z}} H_i(\mathcal{M})$ instead of $H_i(\mathcal{M})\ \forall i$.

A complex $M_{i+1} \xleftarrow{d_i} M_i \xleftarrow{d_{i-1}} M_{i-1}$ is a cochain complex. It has cohomology

$$H^i(\mathcal{M}) = \ker d_i / \operatorname{Im} d_{i-1}.$$

Pre-class Warm-up

Is the following specification a chain complex?

$$\xrightarrow{\alpha} M_0 \xrightarrow{\beta} M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \xrightarrow{\alpha}$$

where $M_i = \mathbb{Z}^2$ for all i,

$$\alpha = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A Yes

B No

This is a cochain complex.

$$\alpha : M \to N$$

A morphisms of chain complexes, or chain map is a diagram

$$\begin{array}{ccccc}
M_{i+1} & \xrightarrow{d_{i+1}} & M_i & \xrightarrow{d_i} & M_{i-1} \\
\downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} \\
N_{i+1} & \xrightarrow{e_{i+1}} & N_i & \xrightarrow{e_i} & N_{i-1}
\end{array}$$

of module homomorphism so that every square commutes $\alpha_{i-1} d_i = e_i \alpha_i \ \forall i$.

Lemma. A chain map f induces homomorphisms of homology groups. If $f: M \to N$ we get a homom. $H_i(f): H_i(M) \to H_i(N)$

Construction of $H_i(f)$:

$f_i(\mathrm{Im}\, d_{i+1}) \subseteq \mathrm{Im}(e_{i+1})$ and

$f_i(\ker d_i) \subseteq \ker e_i$ because

$f_i d_{i+1}(m) = e_{i+1} f_{i+1}(m) \in \mathrm{Im}(e_{i+1})$

If $d_i(m) = 0$ then $f_i(m) \in \ker e_i$ b/c

$e_i f_i(m) = f_{i-1} d_i(m) = 0$.

We define $H_i(f)(m + d_{i+1}(M_{i+1}))$

$= f_i(m) + e_{i+1} N_{i+1}$ where $m \in \ker d_i$

which makes sense and is independent of choice of m.

Example.

$$\begin{array}{ccccccccc}
 & & \mathbb{Z} & & 0 & & \mathbb{Z} & & H_* \\
0 & \to & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} & \mathbb{Z}^2 & \to & 0 \\
 & & \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} & & \\
0 & \to & \mathbb{Z} & \xrightarrow{[2]} & \mathbb{Z} & \xrightarrow{[0]} & \mathbb{Z} & \to & 0 \\
 & & 0 & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z} & & \\
\text{degrees} & & 2 & & 1 & & 0 & &
\end{array}$$

This is a chain map. e.g.

$$[2]\,[1\ \ 1] = [1, 1] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Question: what is the homology of the second complex in degree 2? In degree 1?

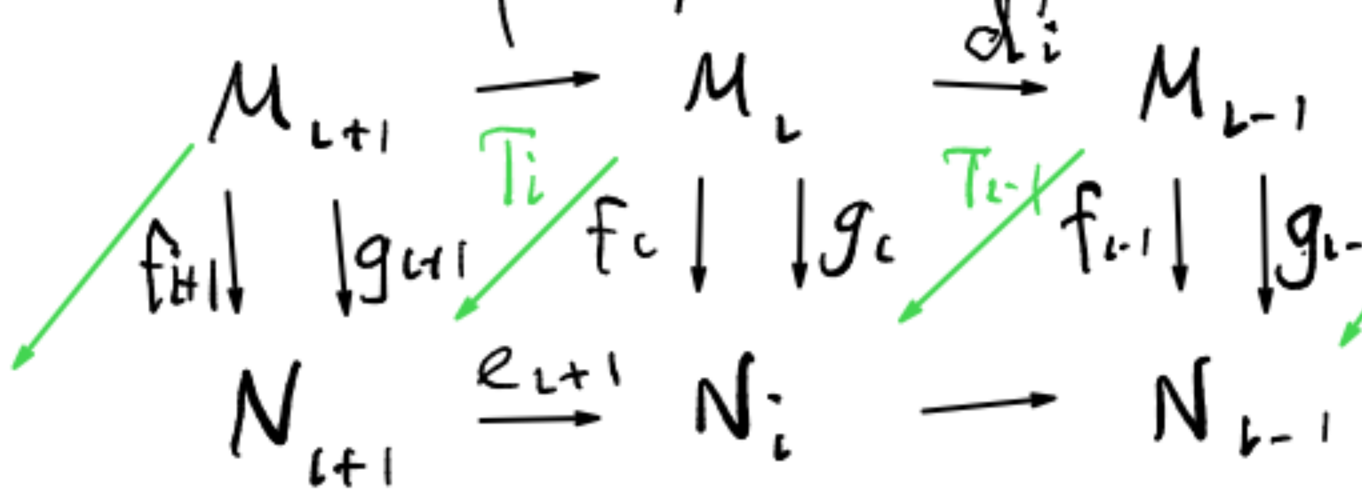
A 0 degree 2

B Z

C Z/2Z degree 1

D something else

Definition. A chain homotopy between two morphisms is specified as follows:



f is homotopic to g, $f \simeq g$

$\iff \exists\ T_i : M_i \to N_{i+1}$ (module homomorphisms) so that

$$f_i - g_i = T_{i-1} d_i + e_{i+1} T_i$$

T is a 'degree $+1$ map'

no commutativity is required.

Proposition. 1. If f and g are homotopic chain maps then the two mappings $H_*(f), H_*(g) : H_*(M) \to H_*(N)$ are the same.

2. If there are chain maps f: $M \to N$ and $g: N \to M$ with $gf \simeq 1_M$ and $fg \simeq 1_N$ then $H_*(f), H_*(g)$ are inverse isomorphisms on homology.

Proof. Suppose $f_i - g_i = T_{i-1} d_i + e_{i+1} T_i$ $\forall i$. Let $m \in \ker d_i \subseteq M_i$

Consider

$f_i(m + \operatorname{Im} d_{i+1}) - g_i(m + \operatorname{Im} d_{i+1})$

$\subseteq (f_i - g_i) m + \operatorname{Im} e_{i+1}$

$= T_{i-1} \underbrace{d_i(m)}_{=0} + e_{i+1} T_i(m) + \operatorname{Im} e_{i+1}$

$= e_{i+1} T_i(m) + \operatorname{Im} e_{i+1} = \operatorname{Im}(e_{i+1})$

$= 0 \in H_i(N)$. Thus $H_i(f) = H_i(g)$

If $f: M \to N$

$g: N \to M$

$gf \simeq 1_M \quad fg \simeq 1_N$

Then $H_*(f): H_*(M) \rightleftarrows H_*(N): H_*(g)$

are inverse isomorphisms.

Proof. $H_*(gf) = H_*(f) H_*(g)$

$= H_*(1_M) = 1_{H_*(M)}$

Also $H_*(g) H_*(f) = 1_{H_*(N)}$.

Thus $H_*(g), H_*(f)$ are inverse isomorphisms. $\square$

Definition Chain complexes M, N are (chain) homotopy equivalent $\iff$ $\exists$ chain maps $f: M \to N$, $g: N \to M$

so that $gf \simeq 1_M$

$fg \simeq 1_N$.

Pre-class Warm-up

Consider the two chain complexes with non-zero terms in degrees 0 and 1:

$$
\begin{array}{rccccccc}
 & & & 0 & & \mathbb{Z}/2\mathbb{Z} & & \\
\mathcal{M}: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 \\
 & {\scriptstyle 0}\downarrow & & {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 0} \\
\mathcal{N}: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \longrightarrow & 0 \\
 & & & 0 & & \mathbb{Z}/3\mathbb{Z} & &
\end{array}
$$

1. Are there any non-zero chain maps $\mathcal{M} \to \mathcal{N}$?

Yes

2. Are there any chain maps $\mathcal{M} \longrightarrow \mathcal{N}$ that induce non-zero maps on homology?

No

A Yes

B No

Examples. k is field

(degrees: 2, 1, 0)

1.
$$\begin{array}{ccccccccccc}
0 & \to & (& k & \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} & k^2 & \xrightarrow{\begin{bmatrix}1 & -1\end{bmatrix}} & k &) & \to 0 \to 0 \\
f\downarrow & & & 1\downarrow & \swarrow{\scriptstyle [0\ 1]} & \downarrow{\scriptstyle \begin{bmatrix}1&0\\0&1\end{bmatrix}} & \swarrow{\scriptstyle \begin{bmatrix}1\\0\end{bmatrix}} & 1\downarrow & & \\
0 & \to & (& k & \xrightarrow[\begin{bmatrix}1\\1\end{bmatrix}]{} & k^2 & \xrightarrow[\begin{bmatrix}1 & -1\end{bmatrix}]{} & k &) & \to 0
\end{array}$$

Consider the degree +1 map T.

Calculate $T_0 d_1 + d_2 T_1$

$$= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= f_1$$

$$d_1 T_0 + T_{-1} d_0 = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = [1] = f_0$$

2. $R = k[x]/(x^3)$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
\alpha\downarrow & 0\downarrow & & \downarrow[x^2] \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
\beta\downarrow & [x^2]\downarrow & & \downarrow 0 \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

$$\begin{array}{cccc}
 & (R & \xrightarrow{[x]} & R) \\
0\downarrow & 0\downarrow & & \downarrow 0 \\
 & (R & \xrightarrow{[x]} & R)
\end{array}$$

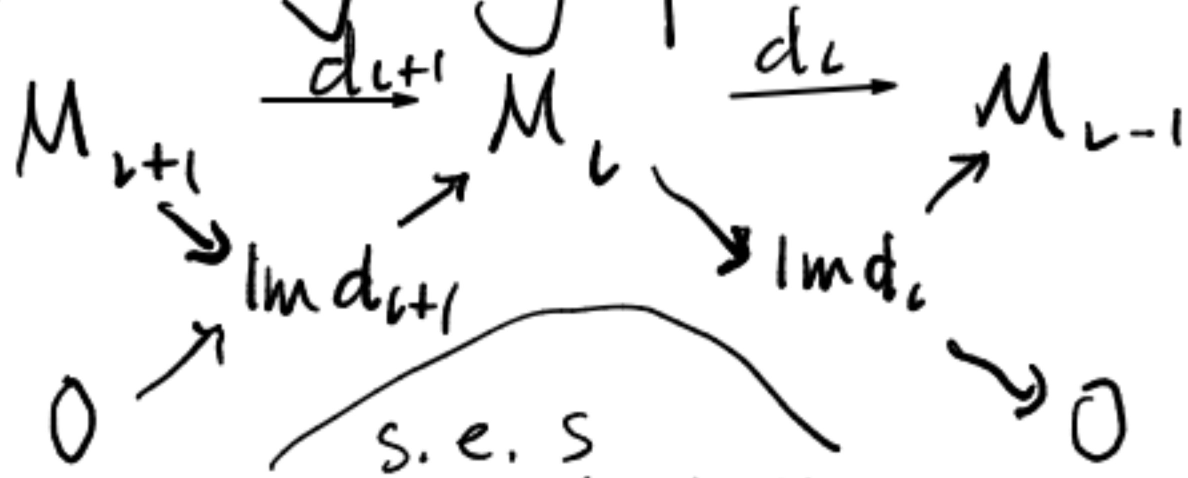
Question. Are any of the chain maps $\alpha, \beta, 0$ chain homotopic? $\alpha \simeq \beta$ $\quad \alpha \not\simeq 0$

$$T_1 d_2 + d_3 T_2 = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1 = f_2.$$

$$f - 0 = Td + dT \text{ so } f \simeq 0.$$

Definition. If the chain complex $\mathcal{M}$ has $1_{\mathcal{M}} \simeq 0$ we call $\mathcal{M}$ contractible.

Dull Fact. $\mathcal{M}$ is contractible $\iff$ it is acyclic (= zero homology = exact everywhere) and "everything splits"



$\iff$ exact at M_i.

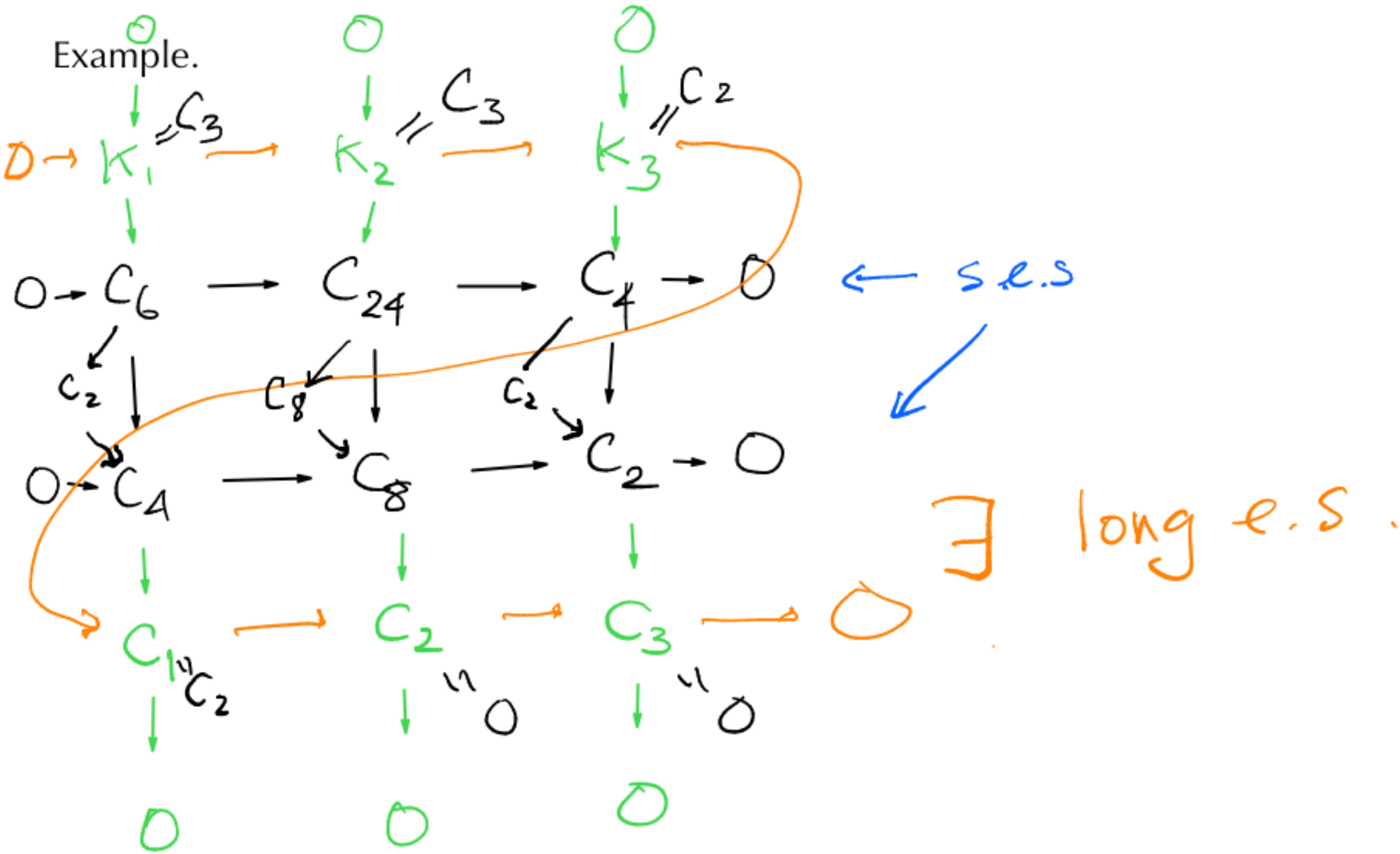
All this s.e.s. split in a contractible cx.

If $X \simeq pt$ is a contractible space then

$1_{\tilde{C}_\cdot(X)}$ is contractible.

The snake lemma

Example.



Pre-class Warm-up!!

Is the following chain complex contractible?

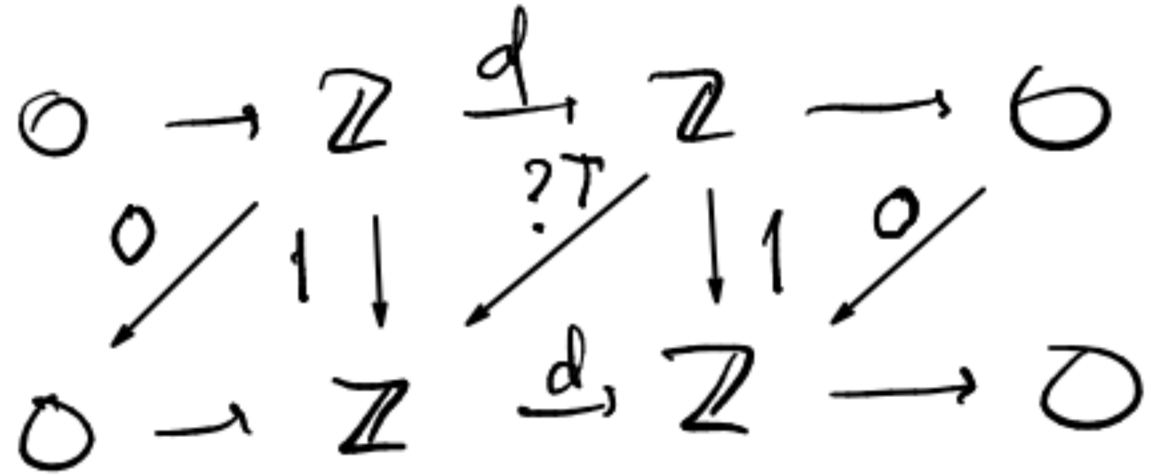
$$0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0$$

A Yes

B No

What about $0 \to \mathbb{Z} \to 0$?

$1_M \simeq 0_M$? Yes:



$1 - 0 = dT + Td$.

Take $T = 1$

Interesting fact:

Contractible complexes are isomorphic to direct sums of shifts of the complex shown, over same ring R:

$$0 \to R \xrightarrow{1} R \to 0$$

The long exact homology sequence.

Theorem 2.3.6. A short exact sequence of chain complexes $0 \to \mathcal{L} \xrightarrow{\Phi} \mathcal{M} \xrightarrow{\Psi} \mathcal{N} \to 0$ gives rise to a long exact sequence in homology.

$$\cdots \xrightarrow{\partial} H_i(\mathcal{M}) \to H_i(\mathcal{N}) \to H_{i-1}(\mathcal{L}) \to H_{i-1}(\mathcal{M}) \to H_{i-1}(\mathcal{N}) \xrightarrow{\partial} H_{i-2}(\mathcal{L}) \to \cdots$$

The connecting homomorphism is natural.

This means: if we have a commutative diagram of s.e.s.

$$\begin{array}{ccccc} (\mathcal{L} & \to & \mathcal{M} & \to & \mathcal{N}) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{L}_1 & \to & \mathcal{M}_1 & \to & \mathcal{N}_1) \end{array}$$

then the squares

$$\begin{array}{ccc} H_i(\mathcal{N}) & \xrightarrow{\partial} & H_{i-1}(\mathcal{L}) \\ \downarrow & & \downarrow \\ H_i(\mathcal{N}_1) & \xrightarrow{\partial} & H_{i-1}(\mathcal{L}_1) \end{array}$$

commute $\forall i$.

Definition. $\mathcal{L} \to \mathcal{M} \to \mathcal{N}$ is exact at $\mathcal{M}$ $\iff$ $\forall i$ $L_i \to M_i \to N_i$ is exact at M_i.

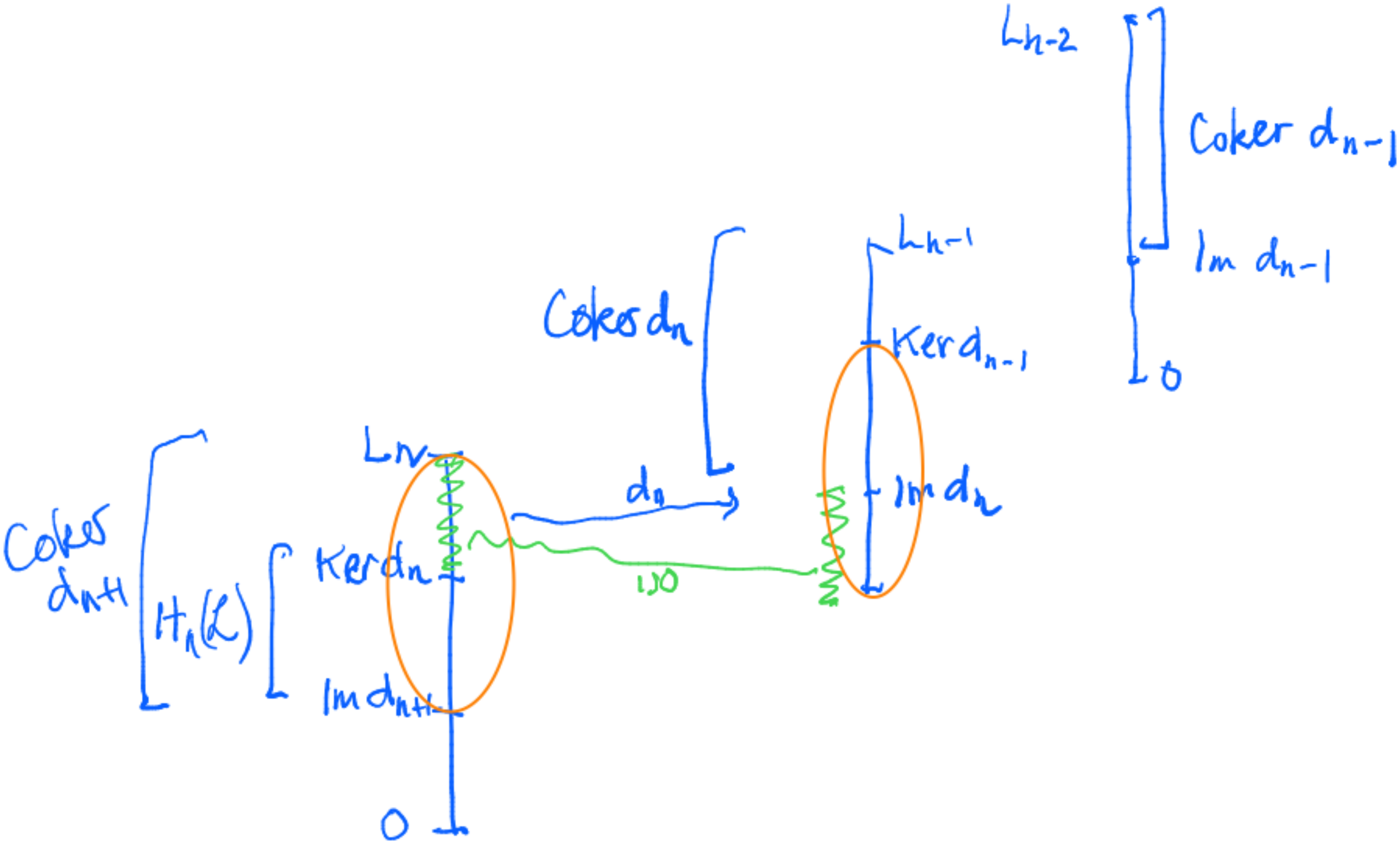
A s.e.s is exact at $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$.

Example. If $X \subseteq Y$ are topological spaces then

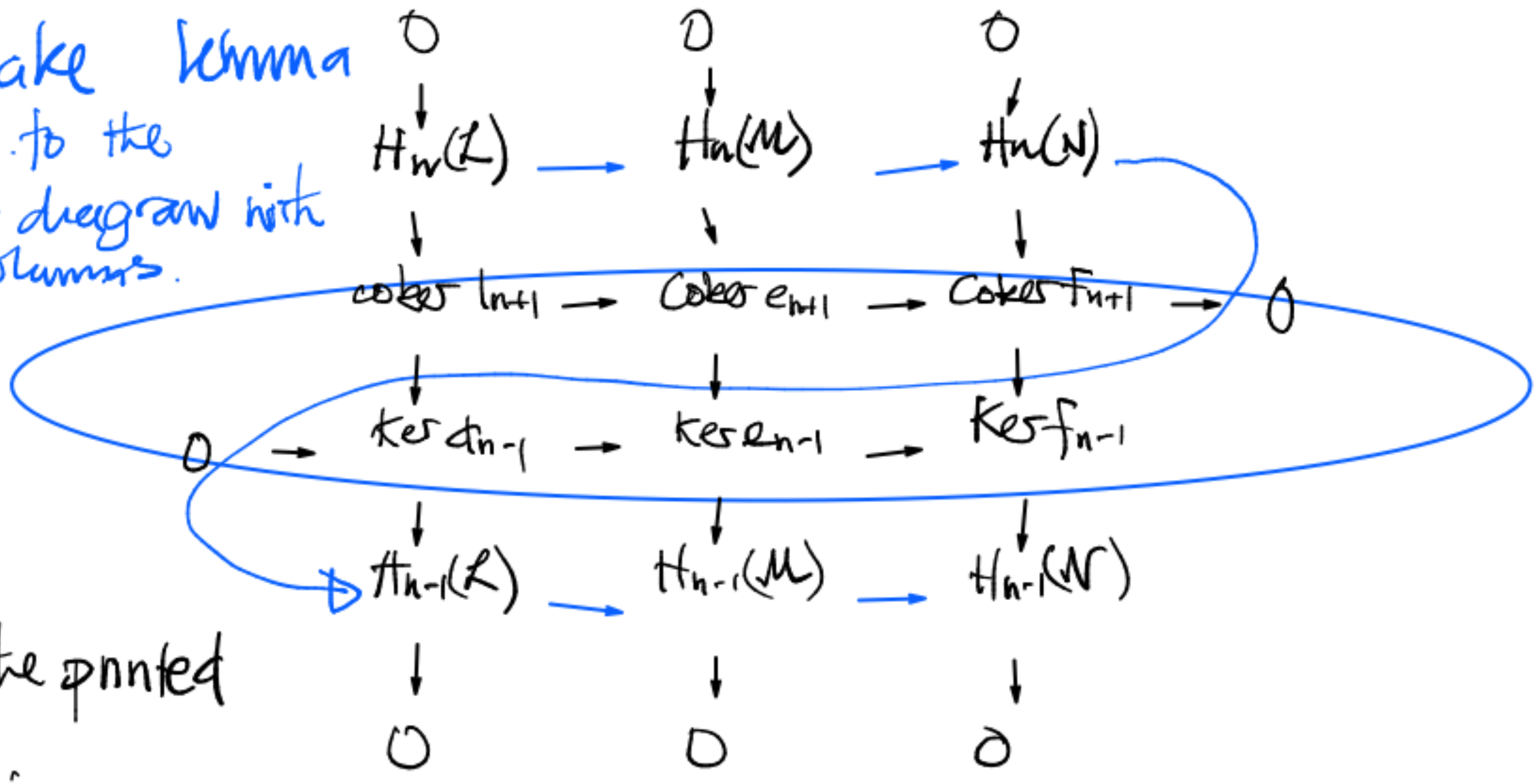
$$0 \to C_\cdot(X) \to C_\cdot(Y) \to C_\cdot(Y,X) \to 0$$

where $C_\cdot(Y,X)$ is defined to be the cokernel of $C_\cdot(X) \to C_\cdot(Y)$.

$\partial : H_i(-) \to H_{i-1}(\ )$ is a natural transformation.

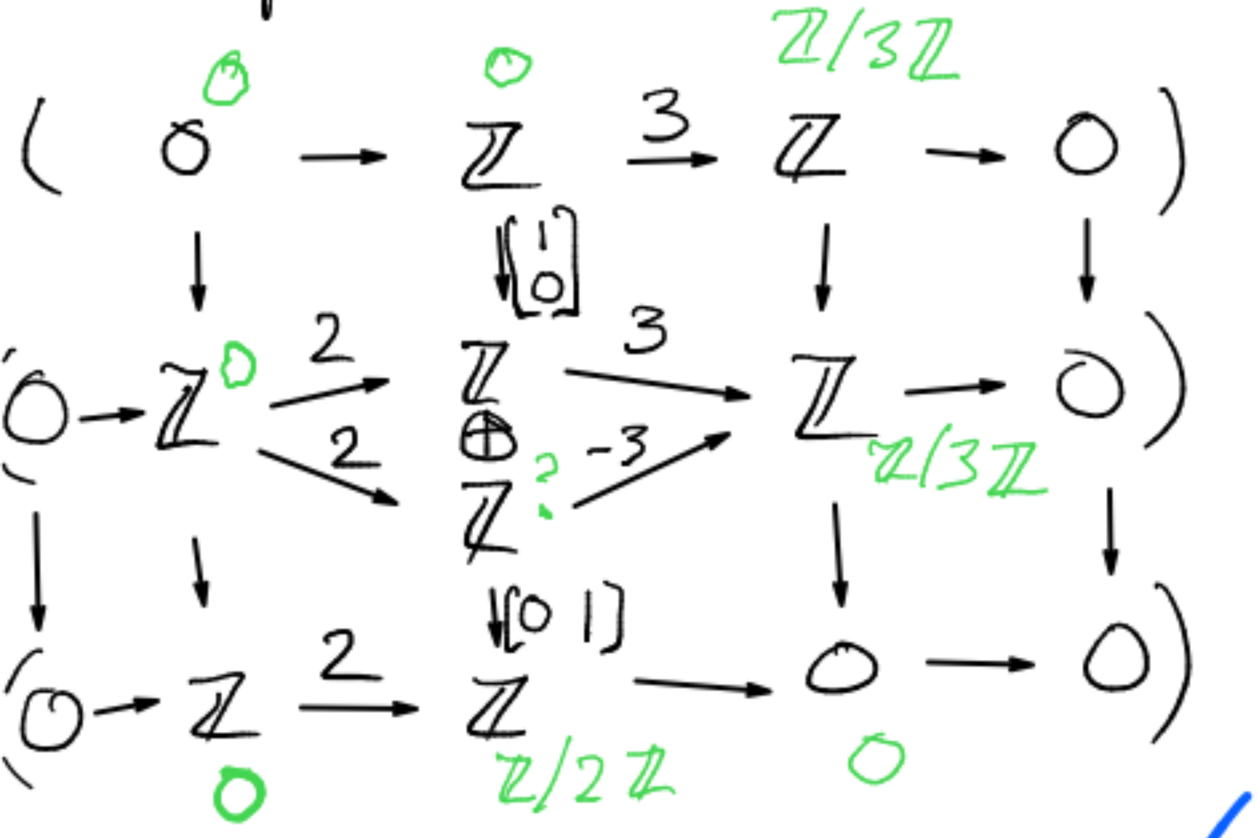

L_{n-2}
Coker d_{n-1}
Im d_{n-1}
0
L_{n-1}
Coker d_n
Ker d_{n-1}
Im d_n
d_n
iso
L_n
Coker d_{n+1}
$H_n(L)$
Ker d_n
Im d_{n+1}
0

The snake lemma applies to the following diagram with exact columns.



See the printed notes.

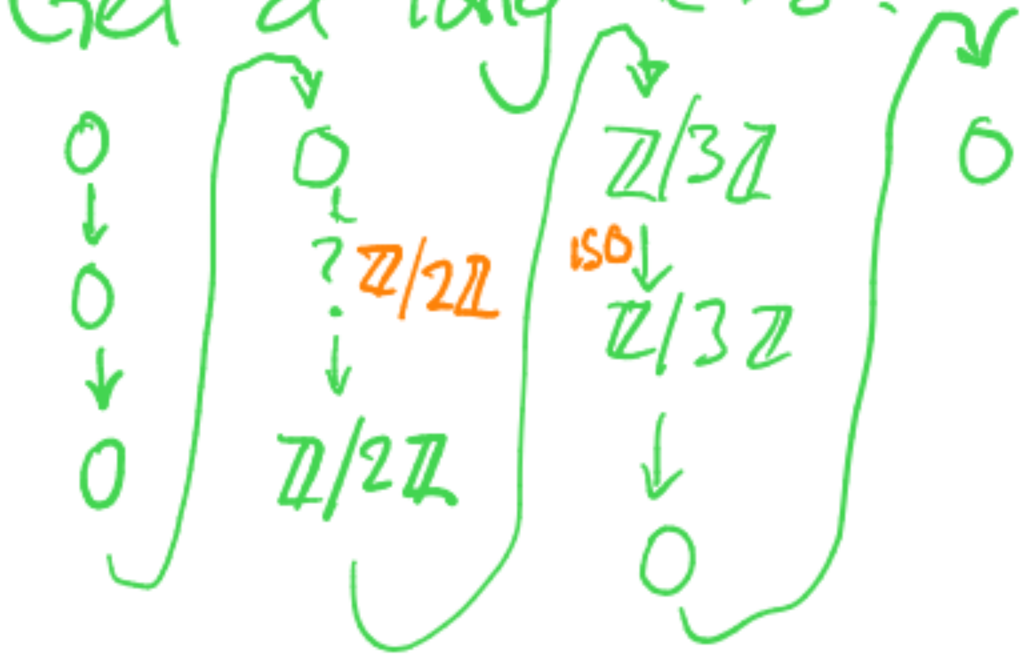
Example:



$0 \to L \to M \to N \to 0$

Check it is a s.e.s. ✓

Get a long e.s.



Pre-class Warm-up!

Let 0 -> A -> B -> C -> 0 be a short exact sequence of modules (for some ring), and let M be some other module. Consider the sequence

0 -> Hom(M,A) -> Hom(M,B) -> Hom(M,C) -> 0 (*)

Which, if any, of the following statements is true?

A If (*) is exact then M is projective.

B If (*) is exact then M is injective.

C If (*) is exact then M is flat.

D If M is projective then (*) is exact. ✓

E If M is injective then (*) is exact.

F If M is flat then (*) is exact.

M is projective $\iff$

$\forall$ $0 \to A \to B \to C \to 0$

(*) is exact.

Projective resolutions

Definition. Let R be a ring and M an R-module. A projective resolution of M is an exact sequence = an acyclic complex

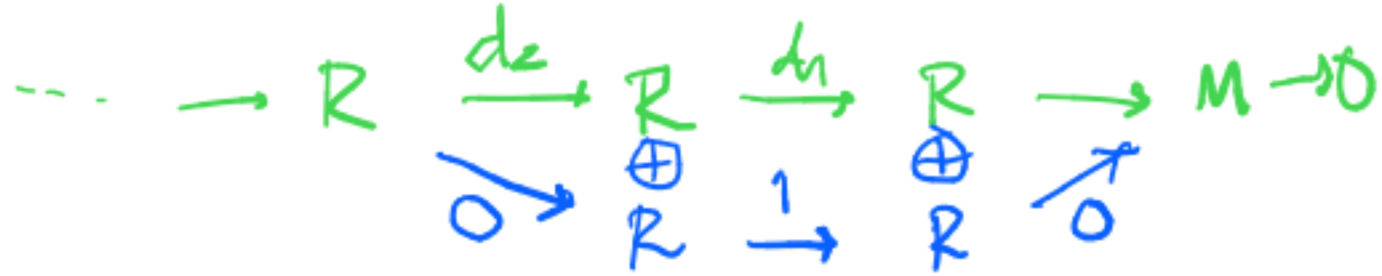
$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

where P_i are projective. We preferably call $\mathcal{P} = \cdots \to P_2 \to P_1 \to P_0 \to 0$ the resolution. It has $H_i(\mathcal{P}) = \begin{cases} M & \text{if } i=0 \\ 0 & \text{o/w} \end{cases}$

Examples. 1. R = Z.

Useful notation: $\mathcal{P} \to M \to 0$ is the projective resolution

A different projective resolution of $(x^3)/(x^5)$:



$= \mathcal{P} \oplus (0 \to R \xrightarrow{1} R \to 0)$

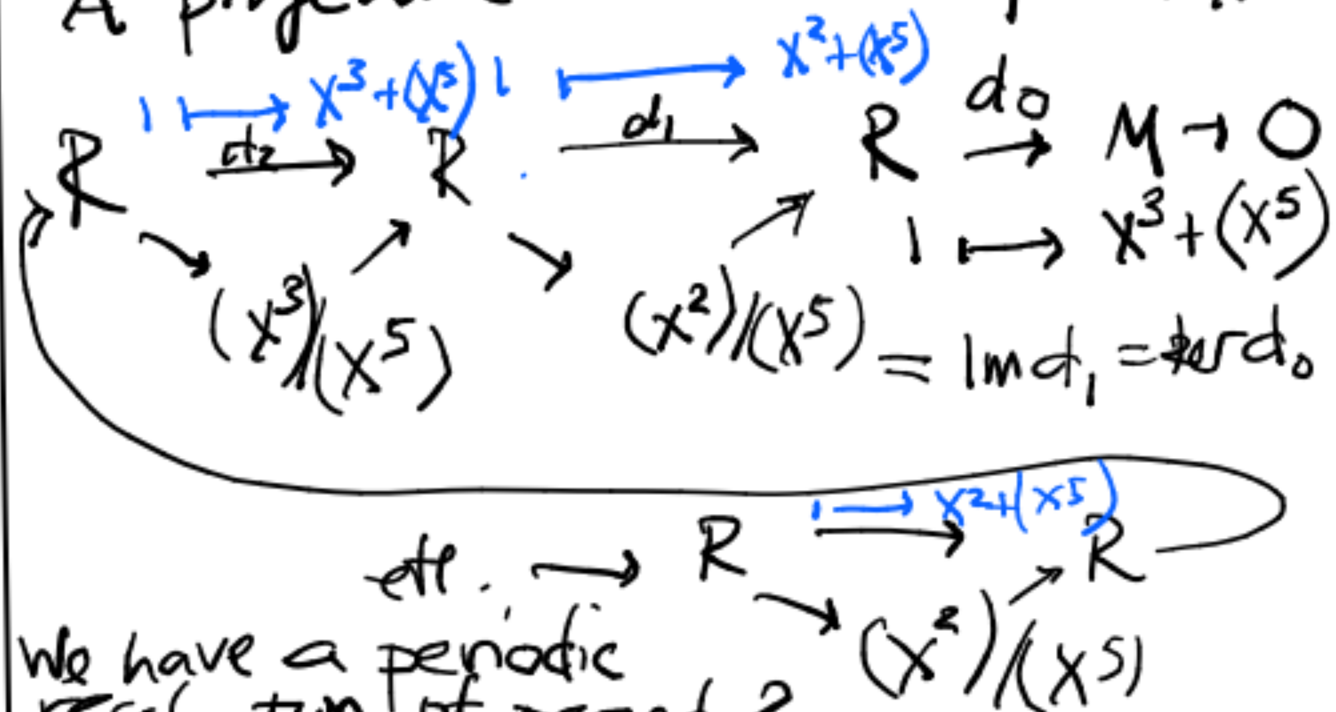
Example. R = k[x]/(x^5) k a field.

R has modules $(x^r)/(x^5)$ = left ideals of R, $0 \le r \le 5$ with submodules

$0 = (x^5)/(x^5) \subseteq (x^4)/(x^5) \subseteq (x^3)/(x^5)$

dim 0 1 2

A projective resolution of $(x^3)/x^5 = M$



$(x^2)/(x^5) = \operatorname{Im} d_1 = \ker d_0$

We have a periodic resolution of period 2.

Question: True or false: all projective resolutions of ~~indecomposable modules~~ M for R have rank 1 projective modules in each position?

A True. B False.

Explanation of $P \to M \to 0$.

We regard the module M as a complex

$$\begin{array}{c|c} \text{degree} & \\ 2 & 0 \\ & \downarrow \\ 1 & 0 \\ & \downarrow \\ 0 & M \\ & \downarrow \\ -1 & 0 \\ -2 & 0 \end{array}$$

called a stalk complex.

We get an exact sequence.

$$P \left[\begin{array}{ccc} P_2 & \longrightarrow & 0 \\ {\scriptstyle d_2}\downarrow & & \downarrow \\ P_1 & \longrightarrow & 0 \\ {\scriptstyle d_1}\downarrow & & \downarrow \\ P_0 & \xrightarrow{d_0} & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \right.$$
$$\begin{array}{ccc} \downarrow & \quad & \downarrow \end{array}$$

Example $R = \mathbb{Z}$. Consider

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \to 0$$

$$\mathbb{Z} = P_1, \qquad \mathbb{Z} = P_0$$

is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$.

Example k is a field

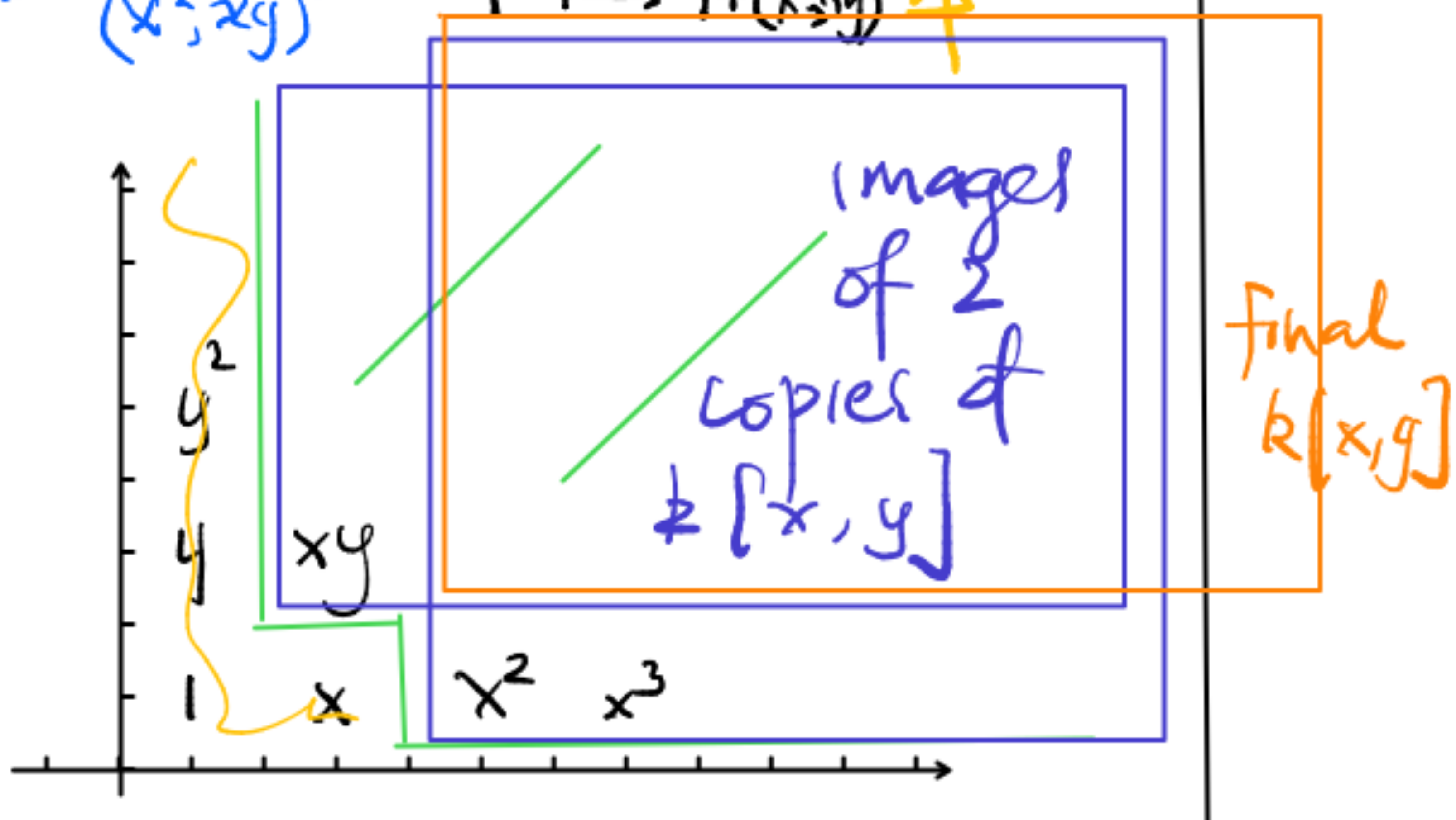
$R = k[x,y]$

$$0 \longrightarrow k[x,y] \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} k(x,y)^2$$

$$\xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} k[x,y] \longrightarrow k[x,y]/(x^2,xy) \to 0$$

(x^2, xy)

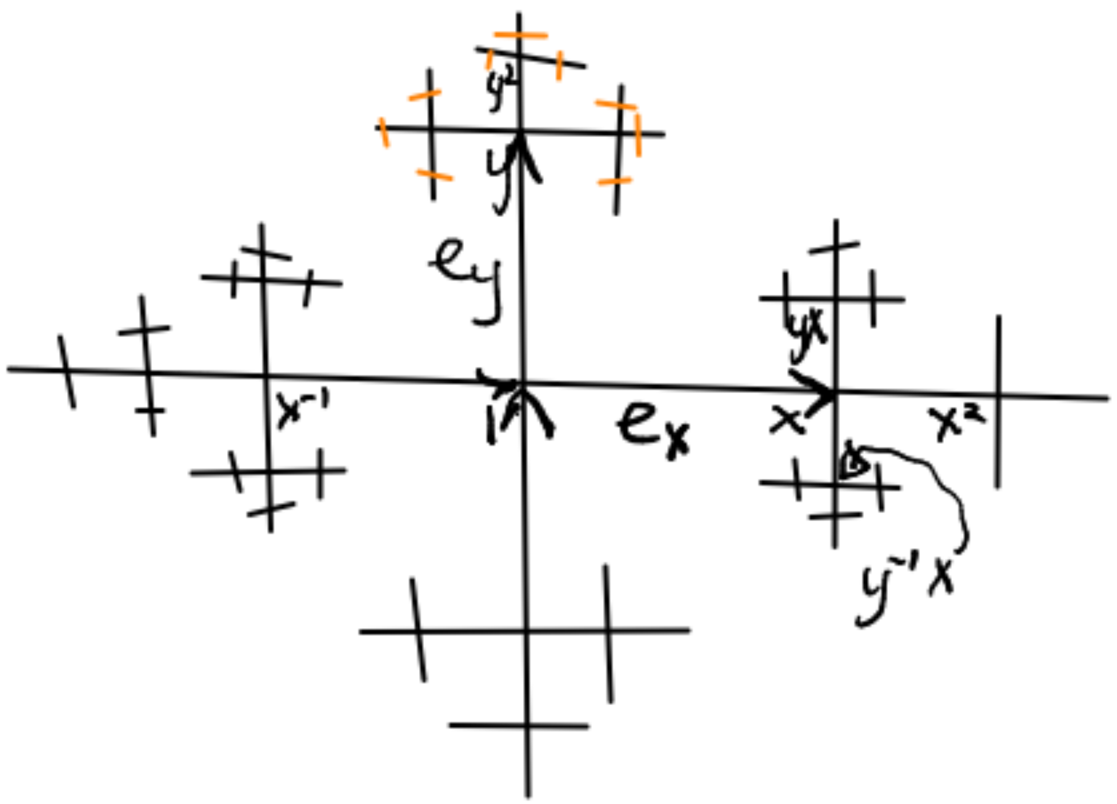
$1 \longmapsto 1 + (x^2, xy)$



Example. Let Z be the integers (~~be a commutative ring~~) and G a group. The group ring ZG is the free $\mathbb{Z}$-module with the elements of G as basis.
Multiplication of basis elements is group multiplication.

Suppose a group G acts freely on a contractible simplicial complex S with chain complex C.(S). Then C.(S) -> Z is a projective ZG-resolution of Z.

Example: G is a free group on 2 generators x, y acting on a tree. Γ



$$0 \to C_1(\Gamma) \xrightarrow{\text{end vertex} - \text{start vertex}} C_0(\Gamma) \to \mathbb{Z} \to 0$$

$C_1(\Gamma)$ = free ... edges $\cong \mathbb{Z}G \oplus \mathbb{Z}G = \mathbb{Z}Ge_x \oplus \mathbb{Z}Ge_y$

$C_0(\Gamma)$ = free abelian group with vertices as basis. $\cong \mathbb{Z}G$

is a projective resolution of $\mathbb{Z}$.

$$0 \to \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Pre-class Warm-up!

Let k be a field and k[x] the polynomial ring in one variable. The ideal (x+1) is a k[x]-module. True or False?:
The k[x]-module (x+1) is projective.

A True ✓

B False

$(x+1) \cong k[x]$ as a $k[x]$-module

$x+1 \longleftrightarrow 1$

$f\cdot(x+1) \longleftrightarrow f$

$(g) \cong k[x] \quad \forall g \in k[x].$

The map k[x]-modules

$k[x] \xrightarrow{g} k[x]$

is 1-1 and has image (g).

degree 1

$0 \to k[x] \xrightarrow{g} k[x] \to k[x]/(g) \to 0$

is a projective resolution of $k[x]/(g)$.

For every PID R, every f.g. module M has a resolution of form

$0 \to R^m \to R^n \to M \to 0.$

$M \cong \bigoplus R/(g)$: take
various $g \in R$

n summands
$n - m = \#$ projective summands

a resolution of each summand and take the $\bigoplus$ of those complexes.

Projective non-free modules

$R = k \oplus k$ with $(a,b)\cdot(c,d)$
$= (ac, bd)$.

$k \oplus 0$ and $0 \oplus k$ are projective modules.

$e_i : R \to k$ ith summand

$1_R = e_1 + e_2 \qquad e_1 e_2 = e_2 e_1 = 0$.

$R = Re_1 \oplus Re_2$ as left modules. Re_1 is projective.

$\mathbb{Z}[e^{2\pi i/23}]$ is not a PID.

All ideals are projective modules

A non principal is not free.

Ext Groups

non-comm. ring

Definition. Let M and N be R-modules.
We define a group $\mathrm{Ext}^n_R(M,N)$ for each $n \geq 0$.
Take a projective resolution

$$P \to M : \underbrace{\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0}_{P} \to M \to 0$$

$$\to 0$$

Form the cochain complex

$$\mathrm{Hom}_R(P,N) = 0 \to \mathrm{Hom}(P_0,N) \to \mathrm{Hom}(P_1,N) \to \mathrm{Hom}(P_2,N) \to \cdots$$

$$\mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}_R(P,N))$$

Example: $\mathbb{Z}/m\mathbb{Z}. = M$ $\quad R = \mathbb{Z}$

Take a projective resolution of M

$$\underbrace{0 \to \mathbb{Z} \xrightarrow{m = d_1} \mathbb{Z}}_{P} \to M \to 0$$

Take an abelian group N.
$\mathrm{Hom}(P,N)$ is

$$0 \leftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},N) \xleftarrow{d_1^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},N) \leftarrow 0$$

$$\quad\ \ \cong \qquad\qquad\qquad\quad \cong$$

$$0 \leftarrow N \xleftarrow{m} N \leftarrow 0$$

$$H^0(\mathrm{Hom}(P,N)) = \mathrm{Ext}^0_{\mathbb{Z}}(M,N) = \ker(m) = \{x \in N \mid mx = 0\}$$

$$H^1(\mathrm{Hom}(P,N)) = \mathrm{Ext}^1_{\mathbb{Z}}(M,N) = N/mN.$$

Example: Z / mZ.

$\mathrm{Ext}^0_{\mathbb{Z}}$ (Z/mZ, N) = { x in N | mx = 0 }

$\mathrm{Ext}^1_{\mathbb{Z}}$ (Z/mZ, N) = N / mN

What are

1. $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$

2. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$

3. $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z})$? $= 0$

4 $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z})$ $= \mathbb{Z}/4\mathbb{Z}$

1. is $\{x \in \mathbb{Z}/6\mathbb{Z} \mid 4 \cdot x = 0\} = \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

2 $\mathbb{Z}/6\mathbb{Z} / 4(\mathbb{Z}/6\mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z} / (\mathbb{Z}/3\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$!

A Z / 2Z

B Z / 4Z

C Z / 6Z

D Z

E 0

Pre-class Warm-up!!

The definition of Ext(M,N) was as follows:

1. Take a projective resolution of M : $\mathcal{P} \to M$
2. Form the cochain complex Hom($\mathcal{P}$, N)
3. Take the degree n cohomology to get $\mathrm{Ext}_R^n(M,N)$

How many things can you find in this definition that seem arbitrary, and could perfectly well have been different, for all we know.?

A 0

B 1

C 2

D 3

E 4

F > 4

Technical

Fact $H^n(\mathrm{Hom}(M, \mathcal{I})) \cong \mathrm{Ext}_R^n(M,N)$.

- Different projective resolutions
- why projective rather than injective or something else.
- why not Hom(N, $\mathcal{P}$)? The technicalities work better.
- why not $\otimes$? We get the groups $\mathrm{Tor}_n^R(M,N)$
- Why not resolve M?
- Why a resolution

$\to P_1 \to P_0 \to M \to 0$

and not a coresolution

$0 \to N \to P_0 \to P_1 \to P_2 \to \cdots$

We can do this with injective modules P_i, then called I_i

$N \to \mathcal{I}$

Proposition 2.4.2 1. $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$

2. If M is projective then $\mathrm{Ext}^i_R(M,N) = 0 \quad \forall i > 0$

Proof 2. Take a projective resolution

$$\underbrace{0 \to M}_{P} \to M \to 0$$

$$\searrow 0$$

Compute $\mathrm{Ext}^n(M,N)$ to get 0 above degree 1.

1. Take a proj. resoln

$$\underbrace{P_1 \xrightarrow{d_1} P_0}_{P} \to M \to 0 \quad (*)$$

$$\searrow 0$$

$\mathrm{Ext}^0_R(M,N) = H^0(\mathrm{Hom}(P,N))$

$\mathrm{Hom}(P,N)$ is

$$0 \to \mathrm{Hom}(P_0,N) \xrightarrow{d_1^*} \mathrm{Hom}(P_1,N) \to$$

$H^0(\mathrm{Hom}(P,N)) = \ker d_1^* / 0$

Apply $\mathrm{Hom}(-,N)$ to $(*)$ to get an exact sequence

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P_0,N) \xrightarrow{d_1^*} \mathrm{Hom}(P_1,N)$$

to see $\ker d_1^* \cong \mathrm{Hom}(M,N)$.

$\square$

Theorem 2.4.3. Let 0 -> A -> B -> C -> 0 be an exact sequence of R-modules and let M be another R-module. There are exact sequences of abelian groups

0 -> Hom(M,A) -> Hom(M,B) -> Hom(M,C)

c.h.

-> Ext^1(M,A) -> Ext^1(M,B) -> ...

connecting homomorphism

$\to \mathrm{Ext}^2(M,A) \to \cdots$

and

0 -> Hom(C,M) -> Hom(B,M) -> Hom(A,M)

c.h.

-> Ext^1(C,M) -> Ext^1(B,M) -> ...

c.h.

$\to \mathrm{Ext}^2(C,M)$

Example: Let C_n be the cyclic group of order n.

Take $0 \to A \to B \to C \to 0$ to be

$0 \to C_2 \to C_4 \to C_2 \to 0$

and $M = C_2$

Can you tell whether the first connecting homomorphism in the long exact sequence is non-zero?

$$0 \to \underset{C_2}{\mathrm{Hom}(C_2,C_2)} \xrightarrow{\text{must be iso}} \underset{C_2}{\mathrm{Hom}(C_2,C_4)} \xrightarrow{\text{must be } 0} \underset{C_2}{\mathrm{Hom}(C_2,C_2)} \xrightarrow[\text{c.h.}]{1\text{-}1}$$

$$\to \mathrm{Ext}^1_{\mathbb{Z}}(C_2,C_2) \to \mathrm{Ext}^1_{\mathbb{Z}}(C_2,C_4) \to \mathrm{Ext}^1(C_2,C_2) \to$$

$$\to \mathrm{Ext}^2(C_2,C_2) = 0$$

A Yes

B No

Can we compute $\mathrm{Ext}^2_R(C_2,C_2) = 0$

Yes

No

Theorem 2.4.3. Let 0 -> A -> B -> C -> 0 be an exact sequence of R-modules and let M be another R-module. There are exact sequences of abelian groups

0 -> Hom(M,A) -> Hom(M,B) -> Hom(M,C)

-> Ext^1(M,A) -> Ext^1(M,B) -> ...

and

0 -> Hom(C,M) -> Hom(B,M) -> Hom(A,M)

-> Ext^1(C,M) -> Ext^1(B,M) -> ...

Proof. 1st long e.s. Take a projective resolution $P \to M$ of M.
We get maps of cochain complexes

$$0 \to \mathrm{Hom}(P,A) \to \mathrm{Hom}(P,B) \to \mathrm{Hom}(P,C) \to 0$$

We show: this is a s.e.s. of cochain complexes.

In each degree this sequence is

$$0 \to \mathrm{Hom}(P_n,A) \to \mathrm{Hom}(P_n,B) \to \mathrm{Hom}(P_n,C) \to 0 \quad (*)$$

$$0 \to \mathrm{Hom}(P_{n+1},A) \to \mathrm{Hom}(P_{n+1},B) \to \cdots$$

$\mathrm{Hom}(P_n,-)$ is exact because P_n is projective.
so (*) is a s.e.s.
Thus the sequence of cochain complexes is exact.
We get a long e.s. □

Pre-class Warm-up!

Should we go through a proof of the long exact sequence in the first variable of Hom? It is the sequence in the yellow box on the right.

A Yes

B No

Theorem 2.4.3. Let 0 -> A -> B -> C -> 0 be an exact sequence of R-modules and let M be another R-module. There are exact sequences of abelian groups

0 -> Hom(M,A) -> Hom(M,B) -> Hom(M,C)

-> Ext^1(M,A) -> Ext^1(M,B) -> …

and

0 -> Hom(C,M) -> Hom(B,M) -> Hom(A,M)

-> Ext^1(C,M) -> Ext^1(B,M) -> …

We have a s.e.s. $0 \to A \to B \to C \to 0$ and get

$$0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M) \to \mathrm{Ext}^1(C,M)$$

We ~~take~~ construct three resolutions

$$\begin{array}{ccc} P' & \xrightarrow{\epsilon'} & A \\ & & \downarrow \\ P & \to & B \\ & & \downarrow \\ P'' & \xrightarrow{\epsilon''} & C \end{array}$$

Construct P', P'' in any way. Form

$$\begin{array}{ccccc} P_0' & \xrightarrow{\epsilon'} & A & & \\ \downarrow \iota_1 & & \downarrow & & \\ P_0' \oplus P_0'' & \xrightarrow{\epsilon} & B & & \\ \downarrow P_2 & \overset{\exists}{\nearrow} & \downarrow & & \\ P_0'' & \xrightarrow{\epsilon''} & C & \to & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

the Δ commutes.

The diagram commutes with component maps $P_0' \oplus P_0'' \to B$ being $(A \to B) \circ \epsilon'$ and lift.

The vertical sequences are s.e.s.

Snake $\Rightarrow$ we get a s.e.s. $0 \to \ker \epsilon' \to \ker \epsilon \to \ker \epsilon'' \to 0$

Repeat the construction with this new s.e.s.

Apply $\mathrm{Hom}(-,M)$ to everything, giving a s.e.s of complexes and a long e.s. in homology.

Corollary 2.4.4.
TFAE for an R-module P.
1. P is projective
2. For all modules M, Ext^1 (P,M) = 0
3. For all modules M, Ext^n (P,M) = 0 for all $n \geq 1$.

P is projective $\iff$ $Ext^1_R(P,M) = 0$ $\forall$ modules M.

Proof. 1 ⇒ 2 and 3: Suppose P is projective. $P = \cdots 0 \to 0 \to P \to 0$ is a projective resolution of P (could draw this as $\underbrace{0 \to P}_{P} \to P \to 0$). It is 0 in degrees ≥ 1, so $Ext^n(P,M) = 0$ if $n \geq 1$.

(2) ⇒ (1) Suppose $Ext^1(P,M) = 0$ $\forall M$. We show $Hom(P,-)$ is exact. Let $0 \to A \to B \to C \to 0$ be a s.e.s. The long e.s. is

$0 \to Hom(P,A) \to Hom(P,B) \to Hom(P,C) \to Ext^1(P,A) \to Ext^1(P,B) \to$, where $Ext^1(P,A) = 0$.

Exact. $Hom(P,-)$ is exact.

Corollary 2.4.4. Part 2.
TFAE for an R-module I.
1. I is injective
2. For all modules M, Ext^1(M,I)=0
3. For all modules M, Ext^n(M,I) = 0 for all $n \geq 1$.

I is injective $\iff$ $Ext^1_R(M,I) = 0$ $\forall M$.

(2) ⇒ (1) is similar to the projective case. Show $Hom(-,I)$ is exact etc.

(1) ⇒ (2) or (3) Suppose I is injective. Compute $Ext^n(M,I)$ using a proj. reso $P \to M$. It is $H^n(Hom(P,I))$. Because $Hom(-,I)$ is an exact functor, the complex $Hom(P,I)$ is acyclic ($n \geq 1$). Thus $Ext^n(M,I) = 0$ if $n \geq 1$.

Dimension shifting

Corollary 2.4.5. Let 0 -> A -> B -> C -> 0 be a short exact sequence of R-modules.

(1) If B is projective then for $n \geq 2$

$$\mathrm{Ext}^n(C,M) \cong \mathrm{Ext}^{n-1}(A,M).$$

(2) If B is injective then for $n \geq 2$

$$\mathrm{Ext}^n(M,A) \cong \mathrm{Ext}^{n-1}(M,C).$$

Proof. 1. Consider the long e.s.

$$\to \mathrm{Ext}^{n-1}(C,M) \to \underbrace{\mathrm{Ext}^{n-1}(B,M)}_{=0} \to \mathrm{Ext}^{n-1}(A,M) \to$$

$$\to \mathrm{Ext}^{n}(C,M) \to \underbrace{\mathrm{Ext}^{n}(B,M)}_{=0} \to \cdots$$

if $n-1 \geq 1$. etc.

2. Is similar.

Pre-class Warm-up

Let k be a field and let R be the ring $R = k[x]/(x^5)$

Let U_2 be the module $U_2 = k[x]/(x^2)$

What is the dimension of $\mathrm{Hom}_R(U_2, R)$?

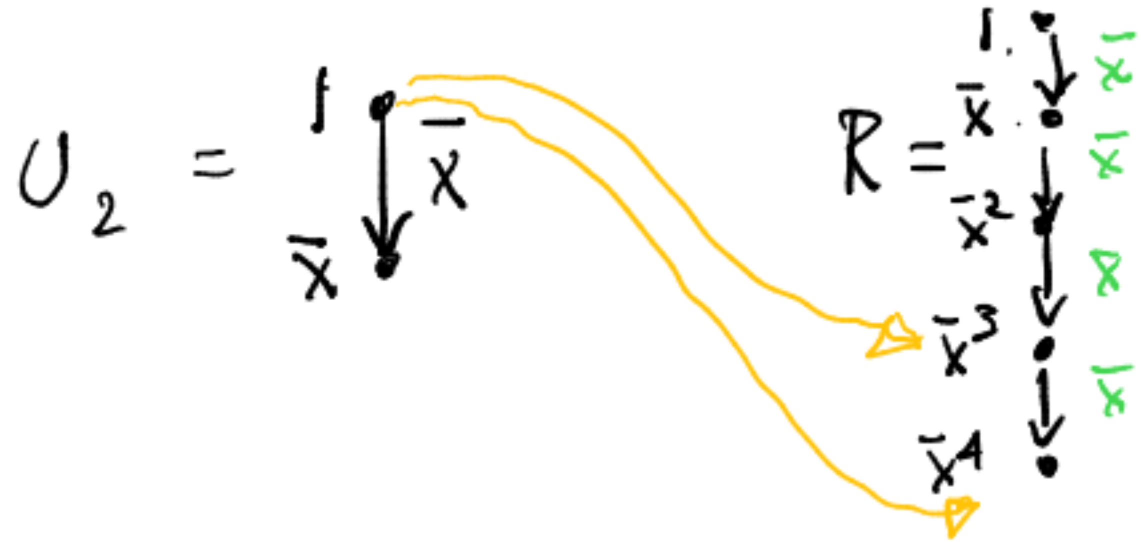
A 0

B 1

C 2 ✓

D 3

E 5



$\bar{x} := x + (x^5) \in R$.

Any homomorphism $U_2 \to R$ is determined by the image of the generator of U_2 : 1 in picture

It must be sent to an element killed by $\bar{x}^2$.

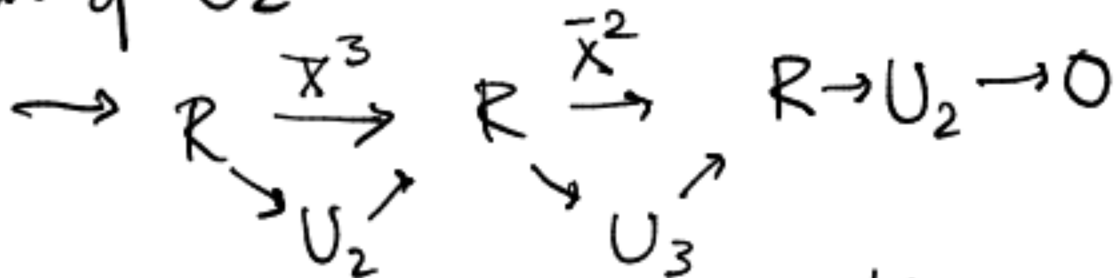
Such elements are the span of $\bar{x}^3, \bar{x}^4$, a space of dimension 2.

$\dim \mathrm{Hom}(U_2, R) = 2$.

Example. $R = k[x]/(x^5)$. This ring has modules

$U_r = (X^{5-r})/(X^5)$ of dimension r. $1 \le r \le 5$

Resolution of U_2



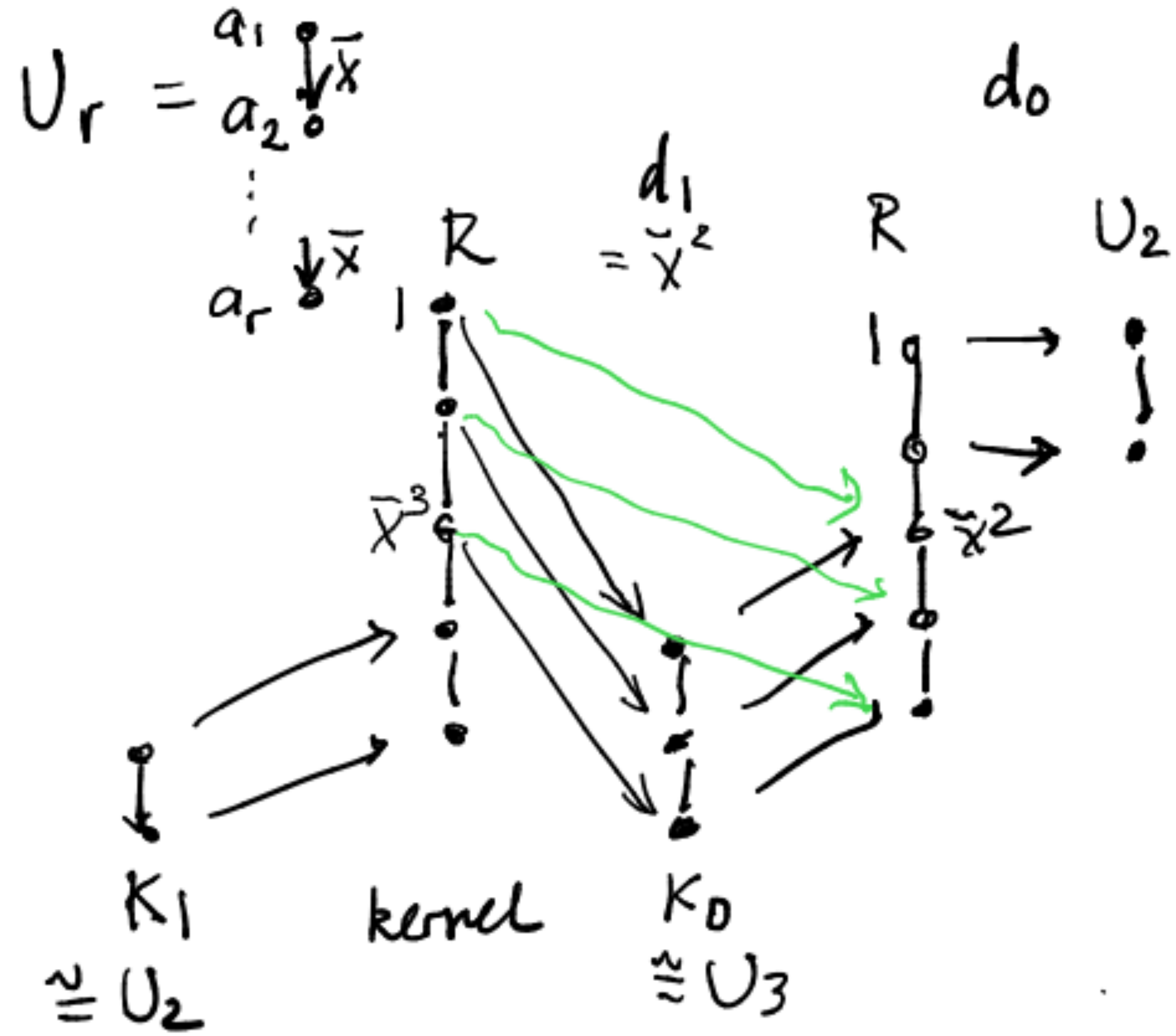
Show

- $\mathrm{Ext}^2(U_2, U_2) \cong \mathrm{Ext}^1(U_3, U_2)$
- $\mathrm{Ext}^1(M, R) = 0 \quad \forall M$. ...
- $\mathrm{Ext}^1(U_2, U_3) \cong \mathrm{Ext}^1(U_3, U_2)$.

Apply 'dimension shifting' using the s.e.s.

$$0 \to U_3 \to R \to U_2 \to 0$$

(R ↑ projective)



$$\mathrm{Ext}^n(U_2, M) \cong \mathrm{Ext}^{n-1}(U_3, M)$$

$\forall$ modules M. $n \geq 2$.

Show $\mathrm{Ext}^1_R(M,R)=0$ $\forall M$.

Take a resolution of M

$$P: \cdots \xrightarrow{\bar{x}^s} R \xrightarrow{\bar{x}^t} R \xrightarrow{\bar{x}^s} R \to M \to 0$$

$s+t=5$.

Compute $\mathrm{Hom}_R(P,R)$

is $\xleftarrow{\bar{x}^s} R \xleftarrow{\bar{x}^t} R \xleftarrow{\bar{x}^s} R$

degree 2 1 0

This complex is acyclic except in degree 0.

$H^i(P,R) = 0$ if $i \geq 1$.

$= \mathrm{Ext}^i_R(M,R)$.

Corollary R is an injective module. (as well as projective).

Pre-class Warm-up!

Which do you think is more successful:

A My use of iPad projected on a screen. 3+2

B Use of blackboard in pre-COVID style. 3

Pros of iPad

I can draw big diagrams in advance

Save & post notes

Technical details

Pros of board

Cons of iPad

Panorama

Pacing of lecture.

Can flip between pages.

in comprehensibly

Which do you think is more successful:

A When I teach entirely online, and everyone is on Zoom.

B When I teach in the classroom in the way I have been doing.

Theorem 2.4.7. Let $\mathcal{P} \to M \to 0$, $\mathcal{Q} \to N \to 0$ be complexes of R-modules, where the modules in $\mathcal{P}$ are projective and

$$\mathcal{Q} \to N \to 0$$

is an acyclic complex.
Every homomorphism ϕ : M -> N lifts to a map of chain complexes

$$\begin{array}{ccc} \mathcal{P} & \to & M \\ \downarrow & & \downarrow \phi \\ \mathcal{Q} & \to & N \end{array}$$

Any two such mappings of complexes that lift ϕ are chain homotopic.

$\mathcal{P} \to M \to 0$ means

$$\begin{array}{ccccccccccc} \cdots \to & P_2 & \to & P_1 & \to & P_0 & \to & M & \to & 0 \\ \exists & \downarrow & & \downarrow & & \downarrow & & \downarrow \phi & & \\ & Q_2 & \to & Q_1 & \to & Q_0 & \to & N & \to & 0 \end{array}$$

a map of chain complexes

Corollary 2.4.8. Let $\mathcal{P}_1 \to M$, $\mathcal{P}_2 \to M$ be two projective resolutions of M.
(1) They are chain homotopy equivalent.
(2) If F is any R-linear functor from R-modules to abelian groups, then
$H_*(F(\mathcal{P}_1)) \cong H_*(F(\mathcal{P}_2))$
by a canonical isomorphism.
(3) Ext^n(M,N) is functorial in both variables.

Proof. Take $\phi = 1$

$$\begin{array}{ccc} \mathcal{P}_1 & \to & M \\ & & \downarrow 1 \\ \mathcal{P}_2 & \to & M \end{array}$$

We get a chain map $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ lifting 1. Similarly we get $\beta : \mathcal{P}_2 \to \mathcal{P}_1$ lifting 1. Now $\beta\alpha : \mathcal{P}_1 \to \mathcal{P}_1$ lifts 1. $\alpha\beta : \mathcal{P}_2 \to \mathcal{P}_2$ lifts 1.
Also $1 : \mathcal{P}_1 \to \mathcal{P}_1$ lifts 1 and $1 : \mathcal{P}_2 \to \mathcal{P}_2$ lifts 1. So
By uniqueness $\beta\alpha \simeq 1_{\mathcal{P}_1}$. $\alpha\beta \simeq 1_{\mathcal{P}_2}$. α, β are inverse chain homotopies. (1) ✓

We have inverse $P_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} P_2$

so that $1_{P_1} - \beta\alpha = Td + dT$

where $T : P_1 \to P_1$ is degree $+1$

d = differential on P_1.

Apply the functor F to get

$$F(1_{P_1} - \beta\alpha) = F(Td + dT)$$

$$1_{FP_1} - F(\beta)F(\alpha) = F(T)F(d) + F(d)F(T)$$ ← shows $F(\beta)F(\alpha) \simeq 1_{F(P_1)}$

$F(d)$ is the differential on $F(P_1)$

$F(T) : F(P_1) \to F(P_1)$ is a degree $+1$ map. ← Similarly $F(\alpha)F(\beta) \simeq 1_{F(P_2)}$

The chain maps $F(P_1) \underset{F(\beta)}{\overset{F(\alpha)}{\rightleftarrows}} F(P_2)$ are inverse up to chain homotopy.

$F(\alpha)_* : H_*(F(P_1)) \to H_*(F(P_2))$ is an isomorphism with inverse $F(\beta)_*$.

(2) ✓

"$\mathrm{Ext}^n(M,N)$ is functorial in both M and N" means given a module homom. $f : N \to L$ we get a group hom. $\mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(M,L)$ so that

To see this:

$$\mathrm{Ext}^n(M,N) = H^n(\mathrm{Hom}(P,N))$$

We get $\mathrm{Hom}(P,N) \to \mathrm{Hom}(P,L)$ a chain map.
hence a map on H^n.

Given a homom $J \to M$ we get a homom of groups $\mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(J,N)$. Take resolutions as follows.

$$\begin{array}{ccccc} Q & \to & J & \to & 0 \\ \downarrow {\scriptstyle \exists \text{ lift}} & & \downarrow & & \\ P & \to & M & \to & 0 \end{array}$$

giving a chain map, $\mathrm{Hom}(P,N) \to \mathrm{Hom}(Q,N)$ hence a morphism on H^n.

Theorem 2.4.7. Let $\mathcal{P} \to M$, $\mathcal{Q} \to N$ be complexes of R-modules, where the modules in $\mathcal{P}$ are projective and

$$\mathcal{Q} \to N \to 0$$

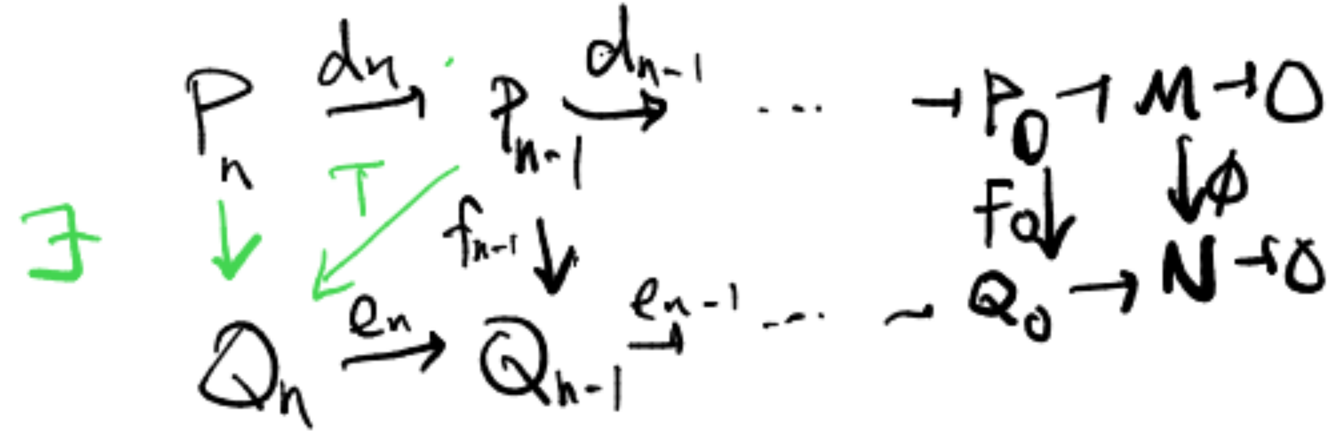
is an a cyclic complex.
Every homomorphism ϕ : M -> N lifts to a map of chain complexes

$$\begin{array}{ccc} \mathcal{P} & \to & M \\ \downarrow & & \downarrow \phi \\ \mathcal{Q} & \to & N \end{array}$$

Any two such mappings of complexes that lift ϕ are chain homotopic.

Proof. We lift inductively. Suppose we have a commutative diagram

we show $f_{n-1} d_n P_n \subseteq \ker e_{n-1}$ in the diagram:



Check $e_{n-1} f_{n-1} d_n = f_{n-2} d_{n-1} d_n$
$= 0 \quad (d^2 = 0)$

We can lift $f_{n-1} d_n : P_n \to Q_{n-1}$ to a map $P_n \to Q_n$. Repeat.

Take two lifts $f, g : \mathcal{P} \to \mathcal{Q}$. We construct $T : \mathcal{P} \to \mathcal{Q}$ of degree +1 so that

$f - g = Td + eT$

$F_0 - g_0$ has image in $e_1(Q_1)$ so lifts to $T_0 : P_0 \to Q_1$, $T_{-1} = 0$
$F_0 - g_0 = e_1 T_0 + T_{-1} d_0$ etc.

A quick summary of Tor

Definition 2.4.9. Let M be a left R-module, N a right R-module, and $P \to N$ a resolution of N by projective right R-modules.

We put $\mathrm{Tor}_n^R(N,M) = H_n(P \otimes_R M)$

Properties:

2.4.10 $\mathrm{Tor}_0^R(N,M) \cong N \otimes_R M$

2.4.11 If 0 -> A -> B -> C -> 0 is a short exact sequence of right R-modules and L is a left R-module, there is a long exact sequence

2.4.13. $\mathrm{Tor}_n^R(N,M) = 0$ if either of M or N is flat and $n > 0$.

2.4.14 Sequence for computing Tor.

$$0 \to \mathrm{Tor}_n^R(N,M) \to K_{n-1} \otimes_R M \to P_{n-1} \otimes_R M \to K_{n-2} \otimes_R M \to 0$$