Chapter 2

Appendix: Basic Homological Algebra

All rings we consider will have a 1, and modules will generally be left unital modules. In this section R may denote any ring. We will need to know about tensor products, and these are described in the books by Dummit and Foote (section 10.4) and Rotman (section 8.4).

Introduce: commutative diagram, category?, monomorphism = injection = mono = 1-1 map

2.1 Tensor products

See Dummit and Foote section 10.4.

Definition 2.1.1. See Dummit and Foote before Theorem 10. If R is a ring, M is a right R-module and N is a left R-module we let X be the free abelian group with basis the elements of $M \times N$ and Y the subgroup generated by all elements of the form $(m_1+m_2, n)-(m_1, n)-(m_2, n)$, $(m, n_1+n_2)-(m, n_1)-(m, n_2)$ and $(mr, n)-(m, rn)$. We define $M \otimes_R N := X/Y$.

Elements of $M \otimes_R N$ are called *tensors*. We write $m \otimes n$ for the image of (m, n) in $M \otimes_R N$, and such tensors are called *simple tensors* or *basic tensors*. Every tensor can be written as a linear combination of simple tensors. In $M \otimes_R N$ the following relations hold:

$$\begin{aligned}(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n\\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2\\ mr \otimes n &= m \otimes rn\end{aligned}$$

We deduce, for example, that $m \otimes 0 = 0 = 0 \otimes n$ for all m and n. From the definition we have that $M \otimes_R N$ has the structure of an abelian group. It does not, in general, have the structure of an R-module.

Definition 2.1.2. Let M be a right R-module, N a left R-module and L an abelian group. A mapping $\phi : M \times N \to L$ is said to be *R-balanced* if and only if

$$\begin{aligned}\phi(m_1 + m_2, n) &= \phi(m_1, n) + \phi(m_2, n)\\ \phi(m, n_1 + n_2) &= \phi(m, n_1) + \phi(m, n_2)\\ \phi(mr, n) &= \phi(m, rn)\end{aligned}$$

always. For example, the mapping $M \times N \to M \otimes_R N$ given by $\phi(m, n) = m \otimes n$ is balanced.

Class Activity. Discuss the difference between the notion of being balanced and some concept of being R-bilinear. We could try to formulate a notion of being R-bilinear using axioms such as the following. Given left R-modules L, M, N, a mapping $\phi : M \times N \to L$ is R-bilinear if and only if

$$\begin{aligned}\phi(r_1 m_1 + r_2 m_2, n) &= r_1\phi(m_1, n) + r_2\phi(m_2, n)\\ \phi(m, s_1 n_1 + s_2 n_2) &= s_1\phi(m, n_1) + s_2\phi(m, n_2)\\ \phi(mr, n) &= \phi(m, rn) = r\phi(m, n).\end{aligned}$$

How much of that makes sense? Is it a problem that $\phi(rm, sn) = r\phi(m, sn) = rs\phi(m, n) = sr\phi(m, n)$?

Theorem 2.1.3 (Dummit and Foote Corollary 11)**.** *The balanced map $M \times N \to M \otimes_R N$ is universal with respect to balanced maps. This means: given a balanced map $M \times N \to L$ there exists a unique group homomorphism $M \otimes_R N \to L$ so that the given balanced map is the composite $M \times N \to M \otimes_R N \to L$. The tensor product $M \otimes_R N$ is defined up to isomorphism by this property.*

Theorem 2.1.4 (Dummit and Foote Theorem 10)**.** *Balanced maps $M \times N \to L$ biject with group homomorphisms $M \otimes_R N \to L$.*

Example 2.1.5. If $f : R \to S$ is a ring homomorphism with $f(1_R) = 1_S$ then $S \otimes_R R \cong S$ as left S-modules via an isomorphism $s \otimes r \mapsto sf(r)$. The left S-module structure comes from multiplication on the left side. Thus, for example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$.

Example 2.1.6. Let I be a right ideal of R. Then $(I\backslash R) \otimes_R M \cong M/IM$. As a proof, we construct inverse maps $(I + r) \otimes m \mapsto rm + IM$ and $(I + 1) \otimes m \leftarrow m + IM$.

Example 2.1.7. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\text{g.c.d.}(m, n)\mathbb{Z}$.

Theorem 2.1.8. *Tensor product distributes over direct sums:*

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N),$$

with a similar formula on the other side.

Proof. This follows from the universal property. □

Example 2.1.9. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$.

Example 2.1.10. Let U and V be vector spaces over a field K with bases $u_1, \ldots, u_r$ and $v_1, \ldots, v_s$. Then the tensors $u_i \otimes v_j$ where $1 \leq i \leq r$ and $1 \leq j \leq s$ form a basis for $U \otimes_K V$.

Sometimes people regard a rank n tensor as an array of numbers $(a_{i,j,k,\ldots})$ with n suffices $i, j, k, \ldots$. Such numbers are the coordinates of the element $\sum a_{i,j,k,\ldots} u_i \otimes v_j \otimes w_k \otimes \cdots$ of the vector space $U \otimes V \otimes W \otimes \cdots$.

Definition 2.1.11. Let $\phi : M \to M'$ and $\psi : N \to N'$ be homomorphisms of right and left R-modules, respectively. We define $\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$ to be the group homomorphism determined by the balanced map $M \times N \to M' \otimes_R N'$ given by $(m, n) \mapsto \phi(m) \otimes \psi(n)$.

Example 2.1.12. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ have matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ have matrix $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, with respect to given bases of $\mathbb{Z}^2$. Then on taking the basis of $\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2$ in a certain order the matrix of $\phi \otimes \psi$ is

$$\begin{bmatrix} 1 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} & 2 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \\ 3 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} & 4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \end{bmatrix}$$

Class Activity. Put the basis vectors $u_i \otimes v_j$ in the correct order so that the above matrix is the matrix of $\phi \otimes \psi$. What is the trace of $\phi \otimes \psi$?

Is base change for rank 2 tensors $B^T a B$ or BAB^{-1}?

Definition 2.1.13. If A and B are rings there is a multiplication on the group $A \otimes B$ defined on basic tensors by $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$, making $A \otimes B$ into a ring.

Examples 2.1.14. Consider exercises 3, 4, 25 of Dummit and Foote. Are any of $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$ isomorphic as rings?

Definition 2.1.15. Let R and S be rings. An (S, R)-bimodule is a left S-module A that is also a right R-module, in such a way that the actions of R and S commute: $(ra)s = r(as)$ for all $r \in R$, $a \in A$ and $s \in S$.

If R is a commutative ring then every left R-module A can also be regarded as a right R-module, and so A is automatically an (R, R)-bimodule. The definition of a bimodule has more serious impact when the rings R and S are not commutative.

If A is an (S, R)-bimodule, B is a left S-module and C is a left R-module then $A \otimes_R C$ is a left S-module with action given by $s(a \otimes c) := sa \otimes c$, $\mathrm{Hom}_S(A, B)$ is a left R-module with action given by $(r\phi)(a) := \phi(ar)$, and $\mathrm{Hom}_S(B, A)$ is a right R-module with action given by $(\phi r)(b) := \phi(rb)$. The operation of tensor product on bimodules is associative.

Theorem 2.1.16 (Dummit and Foote Theorem 43 from 10.5)**.** *Let A be an (S, R)-bimodule, B a left S-module and C a left R-module. Then*

$$\mathrm{Hom}_S(A \otimes_R C, B) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_S(A, B))$$

via an isomorphism that is natural in B and C.

Proof. We define inverse isomorphisms

$$\begin{aligned} \phi &\mapsto (b \mapsto (a \mapsto \phi(a \otimes b))) \\ (a \otimes b \mapsto \psi(b)(a)) &\leftarrow \psi \end{aligned}$$

With the first mapping we check that the image is an R-module homomorphism and that the inner mapping is an S-module homomorphism. With the second mapping we check that it is an S-module homomorphism and that the mapping $(a, b) \to \psi(b)(a)$ is R-balanced.

The two mappings are mutually inverse, and so we have an isomorphism. □

In categorical language, we say that the functor $A \otimes_R - : R\text{-mod} \to S\text{-mod}$ is *left adjoint* to the functor $\mathrm{Hom}_S(A, -) : S\text{-mod} \to R\text{-mod}$, which is *right adjoint* to $A \otimes_R -$.

Corollary 2.1.17. *Let $f : R \to S$ be a ring homomorphism, let B be a left R-module and let C be a left S-module. We regard S as an (S, R)-bimodule where the left action of S is multiplication and the right action of R is multiplication after first applying f. Then $\mathrm{Hom}_S(S \otimes_R B, C) \cong \mathrm{Hom}_R(B, C)$, where C is regarded as a left R-module via the homomorphism f.*

Proof. This is an instance of the previous theorem, because $\mathrm{Hom}_S(S, C) \cong C$ as R-modules via a correspondence $g \leftrightarrow g(1)$. This is an isomorphism of R-modules because if $r \in R$ then $rg \leftrightarrow (rg)(1) = g(r) = r \cdot g(1)$. Note that the action of R on $\mathrm{Hom}_S(S, C)$ is $(rg)(s) = g(sr)$. □

2.2 Splitting and exactness; projective and injective modules

Definition 2.2.1. Let $\alpha : A \to B$ be a homomorphism. We say that α is a *split monomorphism* if there exists a morphism $\beta : B \to A$ so that $\beta\alpha = 1_A$; and we say that α is a *split epimorphism* if there exists a morphism $\beta : B \to A$ so that $\alpha\beta = 1_B$.

Define exact, and short exact sequence.

It is an exercise to see that a split monomorphism is a monomorphism, and a split epimorphism is a epimorphism. From the algebraic point of view of manipulation of symbols, it is a question of identifying whether an element α has a right or left inverse which, in the context of rings, is a natural thing to do. We are also familiar with equivalent conditions for a matrix with entries in a field to have a left or right inverse. Over more general rings the issue is a little more subtle.

Lemma 2.2.2. *Given a short exact sequence of R-modules*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

the following are equivalent:

1. *the monomorphism α is split;*
2. *the epimorphism β is split;*
3. *there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[dr, "i_1"'] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
& & A \oplus C \arrow[u, "\cong"] \arrow[ur, "\pi_2"'] & &
\end{tikzcd}$$

   where i_1 is inclusion and π_2 is projection.

Definition 2.2.3. If any of 1, 2, or 3 of Lemma 2.2.2 is satisfied we say the sequence $0 \to A \to B \to C \to 0$ is *split.*

The next result puts together Theorem 28, Corollary 32, Theorem 33, Proposition 34, Theorem 39 and Corollary 41 from section 10.5 of Dummit and Foote.

Lemma 2.2.4. *Let A, B, C and M be left R-modules, N a right R-module.*

1. *The sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if*
   $0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M)$ *is exact for all M, if and only if*
   $N \otimes_R A \xrightarrow{\alpha_*} N \otimes_R B \xrightarrow{\beta_*} N \otimes_R C \to 0$ *is exact for all N.*
2. *The sequence $0 \to A \to B \to C$ is exact if and only if*
   $0 \to \mathrm{Hom}_R(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}_R(M, B) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, C)$ *is exact for all M.*

Proof. Outline. We first show that if $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact then

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M)$$

is exact. For the converse, assume that

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M)$$

is exact. We show that $B \to C$ is onto: let $B \to C \to C' \to 0$ be exact. Then $0 \to \mathrm{Hom}_R(C', M) \to \mathrm{Hom}_R(c, M) \to \mathrm{Hom}_R(B, M)$ is exact. Therefore $\mathrm{Hom}(C'M) = 0$ for all M, so that $C' = 0$. Next, we show that $\alpha A \subseteq \mathrm{Ker}\,\beta$. If $\beta\alpha \neq 0$ then $\mathrm{Hom}(C, C) to \mathrm{Hom}(A, C)$ maps $1 \to \beta\alpha$ is nonzero. Next we show $\alpha A = \mathrm{Ker}\,\beta$. Take $p : B \to M = B/\alpha A$ in

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M),$$

which has $\alpha^* p = 0$, and $\mathrm{Im}\,\beta^*$ is contained in maps that are zero on $\mathrm{Ker}\,\beta$. Now p is not such unless etc. Use an adjoint property for the $\otimes$? Also, take $N = R$ in one direction. $\square$

This needs some work.

Definition 2.2.5. We say that the functors $\mathrm{Hom}_R(\quad, M)$ and $\mathrm{Hom}_R(M, \quad)$ are *left exact*, while $N \otimes \_\_$ is *right exact*. A covariant functor F is *exact* if and only if whenever $0 \to A \to B \to C \to 0$ is exact then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact, i.e. F is both right and left exact.

Definition 2.2.6. The R-module P is said to be *projective* if and only if given any diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & B \end{array}$$

with α epi there exists $\gamma : P \to A$ such that $\beta = \alpha\gamma$.

Lemma 2.2.7. *The following are equivalent for an R-module P:*

1. *P is projective,*
2. *every epimorphism $M \to P$ splits,*
3. *there is a module Q such that $P \oplus Q$ is free,*
4. $\mathrm{Hom}_R(P, \quad)$ *is an exact functor.*

Definition 2.2.8. There is a similar (dual) definition of an *injective* module. An equivalent condition is that an R-module I is injective if and only if $\mathrm{Hom}_R(\quad, I)$ is an exact functor. Also, an R module N is *flat* if and only if $N \otimes \_\_$ is an exact functor.

Proposition 2.2.9. *Projective modules are flat.*

Proof. Free modules are flat and hence so are projective modules, because they are direct summands of free modules. ☐

2.3 Chain complexes

Definition 2.3.1. A *chain complex* of R-modules is a sequence of R-modules

$$\mathcal{M} = \cdots \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \cdots$$

such that $d_i d_{i+1} = 0$ always. This condition is equivalent to the requirement that $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$ always. We define the *homology group* of $\mathcal{M}$ in degree i to be $H_i(\mathcal{M}) = \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i+1})$. The maps in the family $d = (d_i)$ send modules in given degrees to modules in degree lower by 1, and so we say d has degree -1. We also consider sequences of modules with a family of mappings d of degree $+1$ and in that case we term the sequence a *cochain complex*. The group $H^i(\mathcal{M}) = \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i-1})$ is the *cohomology group* of $\mathcal{M}$ in degree i in this case.

A morphism of complexes $\phi : \mathcal{M} \to \mathcal{N}$ is a sequence of morphisms $\phi_i : M_i \to N_i$ such that

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_3} & M_2 & \xrightarrow{d_2} & M_1 & \xrightarrow{d_1} & M_0 & \xrightarrow{d_0} & \cdots \\ & & \phi_2\downarrow & & \phi_1\downarrow & & \phi_0\downarrow & & \\ \cdots & \xrightarrow{e_3} & N_2 & \xrightarrow{e_2} & N_1 & \xrightarrow{e_1} & N_0 & \xrightarrow{e_0} & \cdots \end{array}$$

commutes. Such a ϕ induces a map $H_n(\phi) : H_n(\mathcal{M}) \to H_n(\mathcal{N})$.

Class Activity. The diagram

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} & \mathbb{Z}^2 & \xrightarrow{\left[\begin{smallmatrix}1&-1\\-1&1\end{smallmatrix}\right]} & \mathbb{Z}^2 \\ \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} & & \downarrow{\scriptstyle [1\ 1]} \\ \mathbb{Z} & \xrightarrow{[2]} & \mathbb{Z} & \xrightarrow{[0]} & \mathbb{Z} \end{array}$$

I'm not sure whether this is a good example. See also the next example

is a morphism of chain complexes. We may compute the homology of a chain complex in general using the Smith normal form for integer matrices. In this example the top complex has homology groups $\mathbb{Z}, 0, \mathbb{Z}$ and the bottom complex has homology groups $0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$.

In different language, a chain complex is a graded R-module $\mathcal{M} = (M_i)_{i\in\mathbb{Z}}$ equipped with a graded endomorphism $d : \mathcal{M} \to \mathcal{M}$ of degree -1 satisfying $d^2 = 0$. This means that d is a module homomorphism and $d(M_i) \subseteq d(M_{i-1})$ for all i. The homology of $\mathcal{M}$ is the graded group $H(\mathcal{M}) = \mathrm{Ker}(d)/\mathrm{Im}(d)$. If the map d had degree $+1$ we would have a *cochain complex* instead.

Definition 2.3.2. A *(chain) homotopy* between two morphisms $\phi, \theta : \mathcal{M} \to \mathcal{N}$ is a graded module morphism $h : \mathcal{M} \to \mathcal{N}$ of degree $+1$ such that $eh + hd = \phi - \theta$. In this case we say that ϕ and θ are homotopic and write $\phi \simeq \theta$.

Proposition 2.3.3. 1. *If ϕ and θ are homotopic then the two mappings $H_n(\phi) = H_n(\theta) : H_n(\mathcal{M}) \to H_n(\mathcal{N})$ are the same.*

2. *If there are chain maps $\phi : \mathcal{M} \to \mathcal{N}$ and $\psi : \mathcal{N} \to \mathcal{M}$ with $\phi\psi \simeq 1_{\mathcal{N}}$ and $\psi\phi \simeq 1_{\mathcal{M}}$ then $H_n(\phi)$ and $H_n(\psi)$ are inverse isomorphisms on homology.*

See Exercise 3 of section 17.1 of Dummit and Foote for the following.

Lemma 2.3.4 (The Snake Lemma)**.** *Let the following commutative diagram of R-modules have exact rows:*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\phi} & B & \xrightarrow{\theta} & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A' & \xrightarrow{\phi'} & B' & \xrightarrow{\theta'} & C' & & \end{array}$$

Then there is an exact sequence

$$\mathrm{Ker}\,\alpha \to \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\gamma \xrightarrow{\omega} \mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma$$

where the mappings between the kernels are the restrictions of ϕ and θ, and the mappings between the cokernels are induced by ϕ' and θ'. Furthermore, if ϕ is mono so is $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta$*, and if θ' is epi so is* $\operatorname{Coker}\beta \to \operatorname{Coker}\gamma$*.*

Proof. The map ω is defined as follows: let $c \in \operatorname{Ker}\gamma$, choose $b \in B$ with $\theta(b) = c$. Then $\theta'\beta(b) = \gamma\theta(b) = 0$ so $\beta(b) = \phi'(a)$ for some $a \in A'$. Define $\omega(c) = a + \alpha(A) \in \operatorname{Coker}(\alpha)$. This is well-defined (see Mr Cooperman's objections in 'It's My Turn'). We now check exactness (see Hilton and Stammbach p.99).

For example, to check exactness at $\operatorname{Ker}\gamma$, we observe first that $\theta(\operatorname{Ker}\beta) \subseteq \operatorname{Ker}\omega$. This is because if $\beta(b) = 0$ then in the construction of $\omega\theta(b)$ we can use the elements $b \in B$, $\beta(b) = 0 \in B'$ and $0 \in A'$, so that $\omega\theta(b) = 0$.

To show that $\theta(\operatorname{Ker}\beta) \supseteq \operatorname{Ker}\omega$ let $c \in \operatorname{Ker}\gamma \cap \operatorname{Ker}\omega$. In constructing $\omega(c)$ we find elements $b \in B$ and $a \in A'$ as above. The element a lies in $\alpha(A)$ because $\omega(c) = 0$. Write $a = \alpha(a_0)$ for some $a_0 \in A$. Now $\beta\phi(a_0) = \phi'\alpha(a_0) = \beta(b)$. Thus $b - \phi(a_0) \in \operatorname{Ker}\beta$ and $\theta(b - \phi(a_0)) = \theta(b) - \theta\phi(a_0) = \theta(b) = c$. Therefore $c \in \theta(\operatorname{Ker}\beta)$.

The remaining arguments are similar. □

Class Activity. Is the morphism of chain complexes given earlier a chain homotopy equivalence? Is the morphism below a chain homotopy equivalence?

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} & \mathbb{Z}^2 & \xrightarrow{\left[\begin{smallmatrix}1&-1\\-1&1\end{smallmatrix}\right]} & \mathbb{Z}^2 \\
\Big\downarrow{\scriptstyle [1\ 0]} & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle [1\ 1]} \\
\mathbb{Z} & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{Z}
\end{array}$$

Try upward morphisms $\begin{bmatrix}1\\0\end{bmatrix}$.

Definition 2.3.5. The mapping ω in the Snake Lemma is called the *connecting homomorphism.* A sequence of complexes $\mathcal{L}\xrightarrow{\phi}\mathcal{M}\xrightarrow{\theta}\mathcal{N}$ is said to be *exact at $\mathcal{M}$* if and only if each for all i, the sequence $L_i\xrightarrow{\phi_i}M_i\xrightarrow{\theta_i}N_i$ of modules in degree i is exact at M_i.

Theorem 2.3.6. *A short exact sequence* $0 \to \mathcal{L}\xrightarrow{\phi}\mathcal{M}\xrightarrow{\theta}\mathcal{N} \to 0$ *of chain complexes gives rise to a long exact sequence in homology:*

$$\cdots \to H_n(\mathcal{L})\xrightarrow{H_n(\phi)}H_n(\mathcal{M})\xrightarrow{H_n(\theta)}H_n(\mathcal{N})\xrightarrow{\omega_n}H_{n-1}(\mathcal{L}) \to \cdots .$$

The connecting homomorphism ω is natural, in the sense that a commutative diagram of chain complexes

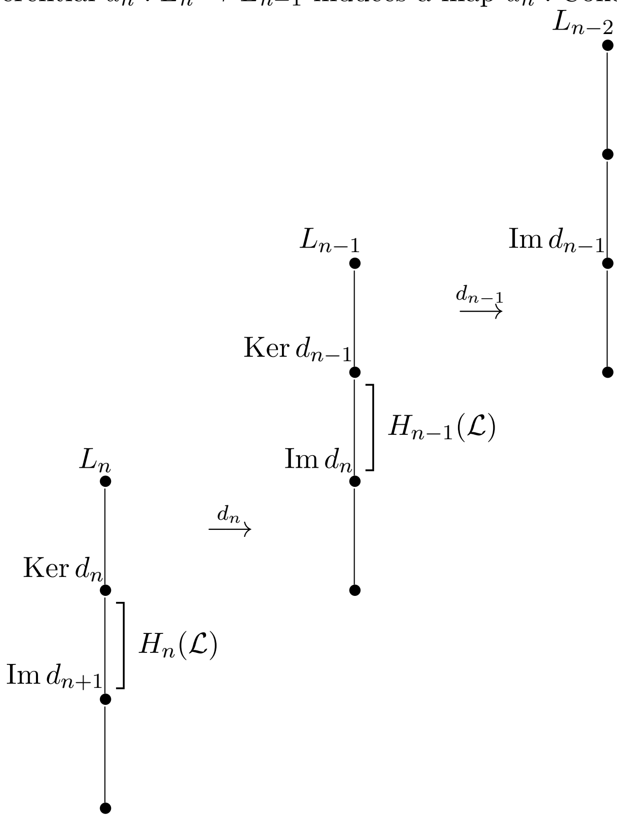
$$\begin{array}{ccccccccc}
0 & \to & \mathcal{L} & \to & \mathcal{M} & \to & \mathcal{N} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L}' & \to & \mathcal{M}' & \to & \mathcal{N}' & \to & 0
\end{array}$$

with exact rows yields a commutative square

$$\begin{array}{ccc} H_n(\mathcal{N}) & \to & H_{n-1}(\mathcal{L}) \\ \downarrow & & \downarrow \\ H_n(\mathcal{N}') & \to & H_{n-1}(\mathcal{L}'). \end{array}$$

Proof. The differential $d_n : L_n \to L_{n-1}$ induces a map $d_n : \operatorname{Coker} d_{n+1} \to \operatorname{Ker} d_{n-1}$:

Label the top two edges of the left term as $\operatorname{Coker} d_{n+1}$.



Similarly with the M's and N's. Apply the snake lemma to the following diagram, all rows and columns of which are exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_n(\mathcal{L}) & & H_n(\mathcal{M}) & & H_n(\mathcal{N}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} d_{n+1} & \longrightarrow & \operatorname{Coker} e_{n+1} & \longrightarrow & \operatorname{Coker} f_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} d_{n-1} & \longrightarrow & \operatorname{Ker} e_{n-1} & \longrightarrow & \operatorname{Ker} f_{n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{n-1}(\mathcal{L}) & & H_{n-1}(\mathcal{M}) & & H_{n-1}(\mathcal{N}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The naturality is an exercise. □

Class Activity. Why are the middle rows of the last big diagram exact? (We use the snake lemma with

$$\begin{array}{ccccccccc}
0 & \to & L_{n+1} & \to & M_{n+1} & \to & N_{n+1} & \to & 0 \\
 & & {\scriptstyle d_{n+1}}\big\downarrow & & {\scriptstyle e_{n+1}}\big\downarrow & & {\scriptstyle f_{n+1}}\big\downarrow & & \\
0 & \to & L_n & \to & M_n & \to & N_n & \to & 0.)
\end{array}$$

Class Activity. Calculate the homology of the kernel complex of the morphism of chain complexes given earlier. Noting that the morphism was surjective in each degree, apply the last theorem with the long exact sequence.

There is a similar result that applies when we have a short exact sequence of cochain complexes $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$. In that case the connecting homomorphism has degree $+1$, giving a long exact sequence

$$\cdots \to H^n(\mathcal{L}) \to H^n(\mathcal{M}) \to H^n(\mathcal{N}) \xrightarrow{\omega_n} H^{n+1}(\mathcal{L}) \to \cdots .$$

2.4 Projective resolutions, Ext and Tor

Let R be a ring and M an R-module. A *projective resolution* of M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in which the P_i are projective modules. Let $\mathcal{P}$ be the complex obtained by replacing M by 0 in the above, so $H_n(\mathcal{P}) = 0$ if $n > 0$ and $H_0(\mathcal{P}) \cong M$ is a given isomorphism. It is useful to write $\mathcal{P} \to M$ to denote this projective resolution.

Example at this point? Maybe $\mathbb{Z}C_2$?

We may always construct resolutions of a module M as follows. Given M, choose a free module P_0 with surjective mapping $P_0 \to M$ and form the kernel K_0. Repeat this process with K_0 instead of M. Depending on the context, other constructions of resolutions may be available: we may have a *bar resolution*, and resolutions constructed from other structures such as a presentation or an action on a space.

Given a second module N we may form the cochain complex

$$\mathrm{Hom}_R(\mathcal{P}, N) = [0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d_0} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_1} \mathrm{Hom}_R(P_2, N) \xrightarrow{d_2} \cdots]$$

obtained by applying $\mathrm{Hom}_R(-, N)$ to $\mathcal{P}$. We now define the degree n Ext group of M and N by

$$\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}_R(\mathcal{P}, N)),$$

the nth cohomology group of this complex.

The above definition depends on the choice of resolution $\mathcal{P}$. It is the case that if we change the resolution we obtain Ext groups that are naturally isomorphic to those just constructed. More of this later!

Example 2.4.1. Let $R = \mathbb{Z}$, so that R-modules are the same thing as abelian groups. For each integer m, the cyclic group $\mathbb{Z}/m\mathbb{Z}$ has a projective resolution as follows:

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

where $\mathcal{P}$ is the chain complex $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$. Taking another abelian group N we compute $\mathrm{Ext}^i(\mathbb{Z}/m\mathbb{Z}, N)$ as the degree i cohomology of the cochain complex

$$\mathrm{Hom}(\mathcal{P}, N) = [\mathrm{Hom}(\mathbb{Z}, N) \xrightarrow{m} \mathrm{Hom}(\mathbb{Z}, N)] = [N \xrightarrow{m} N].$$

Thus

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) \cong \{x \in N \mid mx = 0\}$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) \cong N/mN$$

where $mN = \{mx \mid x \in N\}$. Thus if $N = \mathbb{Z}/p\mathbb{Z}$, where p is prime dividing m, these groups are both $\mathbb{Z}/p\mathbb{Z}$; and if p does not divide m then both groups are 0

Proposition 2.4.2. $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$.

Proof. From the definition, $\mathrm{Ext}^0_R(M, N) = \mathrm{Ker}\, d_0$. Now $P_1 \to P_0 \to M \to 0$ is exact, so

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(P_o, N) \xrightarrow{d_0} \mathrm{Hom}_R(P_1, N)$$

is exact by Lemma 2.2.4, and the result follows. □

Theorem 2.4.3. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules and let M be another R-module. There are exact sequences of abelian groups*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C)& \\ \xrightarrow{\omega} \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(M,B) \to \cdots& \end{aligned} \tag{1}$$

$$\begin{aligned} 0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M)& \\ \to \mathrm{Ext}^1(C,M) \to \mathrm{Ext}^1(B,M) \to \cdots& \end{aligned} \tag{2}$$

Proof. (1) We calculate our Ext groups with a resolution $\mathcal{P} \to M$. The sequence $0 \to A \to B \to C \to 0$ gives a sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(\mathcal{P},A) \to \mathrm{Hom}_R(\mathcal{P},B) \to \mathrm{Hom}_R(\mathcal{P},C) \to 0. \tag{$*$}$$

where, at each level in the grading, this sequence is

$$0 \to \mathrm{Hom}_R(P_n,A) \to \mathrm{Hom}_R(P_n,B) \to \mathrm{Hom}_R(P_n,C) \to 0$$

obtained by applying $\mathrm{Hom}_R(P_n,-)$. Because each P_n is projective, $\mathrm{Hom}_R(P_n,-)$ is exact, and so $(*)$ is a short exact sequence of cochain complexes. We now apply Theorem 2.3.6 and Proposition 2.4.2.

(2) We construct resolutions $\mathcal{P} \to B$, $\mathcal{P}' \to A$ and $\mathcal{P}'' \to C$ appearing in a commutative diagram

$$\begin{array}{ccc} \mathcal{P}' & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ \mathcal{P} & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}'' & \longrightarrow & C \end{array}$$

with exact columns. To do this, let $\mathcal{P}', \mathcal{P}''$ be any resolutions of A and C and construct $\mathcal{P}$ as follows. The start is pictured in a diagram:

$$\begin{array}{ccccc} P_0' & \xrightarrow{\epsilon'} & A & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \\ P_0' \oplus P_0'' & \xrightarrow{\epsilon} & B & & \\ \big\downarrow & & \big\downarrow & & \\ P_0'' & \xrightarrow{\epsilon''} & C & \longrightarrow & 0. \end{array}$$

Lift ϵ'' to a map $P_0'' \to B$ and use this and ϵ' as the components of ϵ, so that the diagram commutes. By the snake lemma, $\mathrm{Ker}\,\epsilon' \to \mathrm{Ker}\,\epsilon \to \mathrm{Ker}\,\epsilon''$ is exact and ϵ is epi. Now repeat this procedure with the terms $\operatorname{Ker}\epsilon' \to \operatorname{Ker}\epsilon \to \operatorname{Ker}\epsilon''$ instead of with $A \to B \to C$, and then with subsequent kernels, to construct $\mathcal{P}''$.

This could be explained better!

Apply $\operatorname{Hom}_R(-, M)$ to this diagram of resolutions and use the fact that

$$0 \to P'_n \to P'_n \oplus P''_n \to P''_n \to 0$$

splits in each degree to get a short exact sequence of cochain complexes

$$0 \to \operatorname{Hom}_R(\mathcal{P}'', M) \to \operatorname{Hom}_R(\mathcal{P}, M) \to \operatorname{Hom}_R(\mathcal{P}', M) \to 0.$$

The long exact sequence in cohomology is the one we are trying to construct. □

Here is an immediate deduction:

Corollary 2.4.4. *1. An R-module P is projective if and only if for all $n \geq 1$ and for all modules M we have $\operatorname{Ext}^n_R(P, M) = 0$.*

2. An R-module I is injective if and only if for all $n \geq 1$ and for all modules M we have $\operatorname{Ext}^n_R(M, I) = 0$.

Proof. (1) If P is projective then $\cdots \to 0 \to P \to P \to 0$ is a projective resolution of P, so that the complex $\operatorname{Hom}_R(\mathcal{P}, M)$ is zero above degree 0 and hence so is its cohomology. Conversely, if $\operatorname{Ext}^n_R(P, M) = 0$ for all $n \geq 1$ then whenever we have a short exact sequence $0 \to A \to B \to C \to 0$ the long exact sequence becomes

$$0 \to \operatorname{Hom}_R(P, A) \to \operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R(P, C) \to \operatorname{Ext}^1_R(P, A) = 0$$

so that $\operatorname{Hom}_R(P, -)$ is an exact functor. It follows that P is projective.

(2) If I is injective then $\operatorname{Hom}_R(-, I)$ is an exact functor so $\operatorname{Hom}_R(\mathcal{P}, I)$ has zero cohomology except in degree 0, and hence the Ext groups are zero above degree 0. Conversely if these Ext groups are zero we deduce as in part (1) from the long exact sequence that $\operatorname{Hom}_R(-, I)$ is an exact functor, so the I is injective. □

We see in the above that we only need the groups $\operatorname{Ext}^1_R(P, M)$ to vanish for all modules M to deduce that P is projective, and similarly only $\operatorname{Ext}^1_R(M, I)$ needs to vanish for all modules M to deduce that I is injective.

Corollary 2.4.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules.*

1. If B is projective then $\operatorname{Ext}^n_R(C, M) \cong \operatorname{Ext}^{n-1}_R(A, M)$ for all modules M, provided $n \geq 2$.

2. If B is injective then $\operatorname{Ext}^{n-1}_R(C, M) \cong \operatorname{Ext}^n_R(A, M)$ for all modules M, provided $n \geq 2$.

Proof. For the proof of 1, part of the long exact sequence becomes

$$0 = \operatorname{Ext}^{n-1}_R(B, M) \to \operatorname{Ext}^{n-1}_R(A, M) \to \operatorname{Ext}^n_R(C, M) \to \operatorname{Ext}^n_R(B, M) = 0$$

giving the claimed isomorphism. The proof of 2 is similar using the long exact sequence in the second variable. □

The process of changing the degree of an Ext group at the expense of changing the module as indicated in the above corollary is known as *dimension shifting*. It is useful in showing that Ext groups are well-defined up to isomorphism, and also in defining operations on the Ext groups, as well as obtaining different identifications of specific Ext groups that arise.

The next result provides a useful way to compute Ext groups.

Proposition 2.4.6. *Let A and M be R-modules, let $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$ be a projective resolution of M, and put $K_i = \operatorname{Ker} d_i$. There is an exact sequence*

$$0 \to \operatorname{Hom}_R(K_{n-2}, A) \to \operatorname{Hom}_R(P_{n-1}, A) \to \operatorname{Hom}_R(K_{n-1}, A) \to \operatorname{Ext}^n_R(M, A) \to 0.$$

Proof. The long exact sequence associated to $0 \to K_{n-1} \to P_{n-1} \to K_{n-2} \to 0$ starts

$$0 \to \operatorname{Hom}_R(K_{n-2}, A) \to \operatorname{Hom}_R(P_{n-1}, A) \to \operatorname{Hom}_R(K_{n-1}, A) \to \operatorname{Ext}^1_R(K_{n-2}, A) \to 0.$$

By dimension shifting we have

$$\operatorname{Ext}^1_R(K_{n-2}, A) \cong \operatorname{Ext}^2_R(K_{n-3}, A) \cong \cdots \cong \operatorname{Ext}^{n-1}_R(K_0, A) \cong \operatorname{Ext}^n_R(M, A).$$

□

One way to prove that Ext groups are well-defined is to prove a corresponding uniqueness statement for projective resolutions, which is what we do now.

Theorem 2.4.7. *Let $\mathcal{P} \to M$ and $\mathcal{Q} \to N$ be complexes of R-modules, where the modules in $\mathcal{P}$ are projective and $\mathcal{Q} \to N \to 0$ is an acyclic complex. Every homomorphism $\phi : M \to N$ lifts to a map of chain complexes*

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & M \arrow[d, "\phi"] \\ \mathcal{Q} \arrow[r] & N \end{tikzcd}$$

and any two such mappings of complexes $\mathcal{P} \to \mathcal{Q}$ that lift ϕ are chain homotopic.

Proof. We construct by induction on n a commutative diagram of the following form, for each n:

$$\begin{tikzcd} P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\phi_{n-1}"] & P_{n-2} \arrow[r, "d_{n-2}"] \arrow[d, "\phi_{n-2}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\phi_0"] & M \arrow[r] \arrow[d, "\phi"] & 0 \\ Q_n \arrow[r, "e_n"] & Q_{n-1} \arrow[r, "e_{n-1}"] & Q_{n-2} \arrow[r, "e_{n-2}"] & \cdots \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

We start the induction at $n = 0$ using projectivity of P_0 and the fact that $Q_0 \to M$ is an epimorphism. For the induction step, suppose that $\phi_0, \ldots, \phi_{n-1}$ have been defined. Now $e_{n-1}\phi_{n-1}d_n = \phi_{n-2}d_{n-1}d_n = 0$, so $\operatorname{Im} \phi_{n-1}d_n \subseteq \operatorname{Ker} e_{n-1} = \operatorname{Im} e_n$. We may now define ϕ_n by the projectivity of P_n.

To show that any two families of maps (ϕ_n) and (ψ_n) lifting ϕ are chain homotopic, we construct mappings $T_n : P_n \to Q_{n+1}$ so that $\phi_n - \psi_n = e_{n+1}T_n + T_{n-1}d_n$ for all $n \geq 0$, with the understanding that $T_{-1} = 0$. We define $\phi_{-1} = \psi_{-1} = \phi$. Suppose that T_{n-1} has been constructed. We calculate

$$\begin{aligned} e_n(\phi_n - \psi_n - T_{n-1}d_n) &= \phi_{n-1}d_n - \psi_{n-1}d_n - e_nT_{n-1}d_n \\ &= (\phi_{n-1} - \psi_{n-1} - e_nT_{n-1})d_n \\ &= T_{n-2}d_{n-1}d_n \\ &= 0. \end{aligned}$$

Therefore $\operatorname{Im}(\phi_n - \psi_n - T_{n-1}d_n) \subseteq \operatorname{Im} e_{n+1}$ and so there exists T_n with

$$(\phi_n - \psi_n - T_{n-1}d_n) = e_{n+1}T_n,$$

by projectivity of P_n. Rearranging this equation, it is $\phi_n - \psi_n = e_{n+1}T_n + T_{n-1}d_n$, as required. □

Corollary 2.4.8. *Let $\mathcal{P}_1 \to M$ and $\mathcal{P}_2 \to M$ be two projective resolutions of M.*
(1) $\mathcal{P}_1 \to M$ and $\mathcal{P}_2 \to M$ are chain homotopy equivalent.
(2) If F is any R-linear functor from R-modules to abelian groups, then

$$H_*(F(\mathcal{P}_1) \cong H_*(F(\mathcal{P}_2)$$

by a canonical isomorphism.
(3) $\operatorname{Ext}^n_R(M, N)$ is functorial in both variables.

We remark also that $Ext^n_R(M, N)$ can also be defined by taking an injective resolution $N \to \mathcal{I}$ of N and forming $H_n(\operatorname{Hom}_R(M, \mathcal{I}))$. It is a theorem that we get a group that is naturally isomorphic to the group defined by a projective resolution of M. We say that Ext is *balanced* to indicate that it has this property.

Definition 2.4.9. Let M be a left R-module, N a right R-module, and $\mathcal{P} \to N$ a resolution of N by projective right modules. We put

$$\operatorname{Tor}^R_n(N, M) = H_n(\mathcal{P} \otimes_R M),$$

which is the nth homology of the complex

$$\cdots \to P_2 \otimes_R M \to P_1 \otimes_R M \to P_0 \otimes_R M \to 0.$$

Tor has properties analogous to those of Ext and we list them below. They are proved in a similar manner to the corresponding results for Ext, using that $\_\_ \otimes_R M$ is right exact instead of left exact.

Proposition 2.4.10. $\operatorname{Tor}^R_0(N, M) \cong N \otimes_R M$*.*

Theorem 2.4.11. *If $0 \to A \to B \to C \to 0$ and $0 \to L \to M \to N \to 0$ are short exact sequences of right and left modules respectively there are long exact sequences*

$$(i) \qquad \begin{aligned} &\cdots \to \mathrm{Tor}_2^R(C, L) \to \mathrm{Tor}_1^R(A, L) \to \mathrm{Tor}_1^R(B, L) \to \mathrm{Tor}_1^R(C, L) \\ &\quad \to A \otimes_R L \to B \otimes_R L \to C \otimes_R L \to 0 \end{aligned}$$

and

$$(ii) \qquad \begin{aligned} &\cdots \to \mathrm{Tor}_2^R(A, N) \to \mathrm{Tor}_1^R(A, L) \to \mathrm{Tor}_1^R(A, M) \to \mathrm{Tor}_1^R(A, N) \\ &\quad \to A \otimes_R L \to A \otimes_R M \to A \otimes_R N \to 0. \end{aligned}$$

Remark 2.4.12. One can view Tor as a measure of the failure of $\otimes$ to be left exact.

Proposition 2.4.13. $\mathrm{Tor}_n^R(N, M) = 0$ *if either of M or N is flat and $n > 0$.*

It follows that $\mathrm{Tor}_n^R(N, M) = 0$ if M or N is projective, because projective modules are flat. This allows a process of 'dimension shifting' analogous to that for Ext.

In the next result we let

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2"] \arrow[dr] & & P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \arrow[r] & N \arrow[r] & 0 \\
& & K_1 \arrow[ur] & & K_0 \arrow[ur] & & &
\end{tikzcd}$$

be the resolution of N, so that $K_n = d_{n+1}(P_{n+1})$.

Proposition 2.4.14. *There is an exact sequence*

$$0 \to \mathrm{Tor}_n^R(N, M) \to K_{n-1} \otimes_R M \to P_{n-1} \otimes_R M \to K_{n-2} \otimes_R M \to 0$$

for $n \geq 1$. (Here we take $K_{-1} = N$.)

Remark 2.4.15. We can also calculate $\mathrm{Tor}_n^R(N, M)$ by taking a projective resolution of M by left modules, applying $N \otimes_R -$ and taking homology of the resulting complex. In this way one obtains a sequence of functors that turn out to be naturally isomorphic to the functors we have defined.

2.5 Pushouts, pullbacks and Schanuel's lemma

For this see section 10.5 of Dummit and Foote, Exercises 27 and 28.